

CHAPTER 10: SYMBOLIC TRAILS AND FORMAL PROOFS OF VALIDITY, PART 2

Introduction

In the previous chapter there were many frustrating signs that something was wrong with our formal proof method that relied on only nine elementary rules of validity. Very simple, intuitive valid arguments could not be shown to be valid. For instance, the following intuitively valid arguments cannot be shown to be valid using only the nine rules.

Somalia and Iran are both foreign policy risks. Therefore, Iran is a foreign policy risk.

$$\begin{array}{l} S \bullet I \\ \hline \therefore I \end{array}$$

Either Obama or McCain was President of the United States in 2009.¹

McCain was not President in 2010. So, Obama was President of the United States in 2010.

$$\begin{array}{l} (O \vee C) \bullet \sim(O \bullet C) \\ \sim C \quad \hline \therefore O \end{array}$$

If the computer networking system works, then Johnson and Kaneshiro will both be connected to the home office. Therefore, if the networking system works, Johnson will be connected to the home office.

$$\begin{array}{l} N \supset (J \bullet K) \\ \hline \therefore N \supset J \end{array}$$

Either the Start II treaty is ratified or this landmark treaty will not be worth the paper it is written on. Therefore, if the Start II treaty is not ratified, this landmark treaty will not be worth the paper it is written on.

$$\begin{array}{l} R \vee \sim W \\ \hline \therefore \sim R \supset \sim W \end{array}$$

¹ This *or* statement is obviously exclusive, so note the translation.

If the light is on, then the light switch must be on. So, if the light switch is not on, then the light is not on.

$$\begin{array}{l} L \supset S \\ \therefore \sim S \supset \sim L \end{array}$$

Thus, the nine elementary rules of validity covered in the previous chapter must be only part of a complete system for constructing formal proofs of validity. A **complete** system must allow us to be able to make a formal proof for any valid argument that can be translated into propositional logic form, the level of argumentation consisting of simple statements and the five logical connectives. So, to complete our system 10 more elementary arguments are needed. The first 9 rules we will call **Rules of Inference**, the second 10 are called **Rules of Replacement**. Although these new replacement rules involve the same level of abstraction, they differ somewhat in application; we will have much more flexibility in applying them. For instance, a rule of inference such as modus ponens can be applied to only whole lines and the direction of inference is only in one direction. The premises cannot be derived from the conclusion. On the other hand, a rule of replacement can be applied to a whole line or part of a line, and the direction of inference can go in two directions.

For instance, **De Morgan's theorem** is an example of a rule of replacement: $\sim(p \bullet q) \equiv (\sim p \vee \sim q)$. This rule tells us that any statement or part of a statement that has the form of $\sim(p \bullet q)$ can be replaced with $(\sim p \vee \sim q)$, and any statement or part of a statement that has the form of $(\sim p \vee \sim q)$ can be replaced with $\sim(p \bullet q)$. Whereas a truth table of modus ponens shows the argument form to be valid, a truth table shows De Morgan's theorem to be a **logical equivalence**, the right hand side of the $(\equiv)$ symbol has the same truth table result as the left hand side.

p	q	$\sim(p \bullet q)$				$\equiv$	$(\sim p \vee \sim q)$		
T	T	F	T	T	T	T	F	F	F
T	F	T	T	F	F	T	F	T	T
F	T	T	F	F	T	T	T	T	F
F	F	T	F	F	F	T	T	T	T

Note that when $\sim(p \bullet q)$ is false (1st row), $\sim p \vee \sim q$ is also false. When $\sim(p \bullet q)$ is true (rows 2-4), $\sim p \vee \sim q$ is also true. Because a biconditional ($\equiv$) is true when the truth values of both sides match, the truth value result of De Morgan's theorem is always true. **A logical equivalence is a biconditional that is always true.** All of the new rules of replacement will be logical equivalences, which justifies our using them in manipulating statements in symbolic reasoning trails.

At a less formal level of justification, it should help you to understand that all the new rules are, like the nine rules of inference, just formalizations of our common sense. We have already been using De Morgan's theorem when we determined in Chapter 7 that saying that, "Alice and

Barbara are not both coming to the party" $\sim(A \bullet B)$, is the same as saying that, "Either Alice is not coming to the party or Barbara is not coming to the party." $(\sim A \vee \sim B)$ Regardless of how threatening they may look at first, keep in mind that all the new rules originate from our common sense, and that we are still following the same procedure of breaking our common sense down into very simple rules, so that we can use them to create chains of reasoning and analyze complex arguments.

At a practical level of proof construction, these new rules give us considerable flexibility. Because a rule of replacement can be applied to whole or parts of statements, and the inference direction can be in either direction of the $(\equiv)$ symbol, there are four different possible applications for each rule of replacement. For instance, here are four different ways that the De Morgan rule could be applied. Because $\sim(p \bullet q) \equiv (\sim p \vee \sim q)$, we could have:

- | | |
|--|--|
| 1. $\sim(A \bullet B) \therefore \sim A \vee \sim B$ | 3. $\sim(A \bullet B) \supset C \therefore (\sim A \vee \sim B) \supset C$ |
| 2. $\sim A \vee \sim B \therefore \sim(A \bullet B)$ | 4. $(\sim A \vee \sim B) \supset C \therefore \sim(A \bullet B) \supset C$ |

Notice that in 1 and 2 the rule is applied to whole lines, but operates in either direction; whereas in 3 and 4 the rule is applied to a part of a line, and also operates in either direction. Because De Morgan's theorem is a logical equivalence, we are entitled to "replace" any statement or part of a statement that fits the form of one side of the biconditional with its equivalent on the other side. This replacement capability will give us a great deal of flexibility in deriving proofs. To help you remember this flexibility, students often write the new rules down with a double arrow.

$$\sim(p \bullet q) \leftrightarrow (\sim p \vee \sim q)$$

However, a major restriction that must never be violated is that ***if a rule of replacement is applied to part of a line, the whole line must be concluded***. Notice that in examples 3 and 4, the conclusion included the consequent C. The conclusion must always be equivalent to the premise when applying a rule of replacement. If the consequent was dropped, then the conclusions in both examples would not be equivalent to the premises. The following would not be valid and would not be an application of De Morgan's theorem:

$$(\sim A \vee \sim B) \supset C \therefore \sim(A \bullet B) \quad \mathbf{X}$$

Below is a complete list of all nineteen rules that we will be using to construct formal proofs. This system is one of many possible complete systems. Most students find this list overwhelming and quite abstract at first glance, and it takes some time to adjust to applying the new rules just as it did with the first nine. However, keep in mind that each of the new rules has a common sense equivalent and that the process of pattern recognition is the same as with the first nine rules. We will also gradually become more comfortable using these rules by going

through the same three steps (Steps 4-6) as with the nine rules. Most important for your success at this stage is that if you have been successful with the nine rules (and you shouldn't be on this page if you have not, go back to a previous step in Chapter 9), then you understand the symbolic game: regardless of how esoteric, abstract, and complicated a symbolic statement or transformation may seem, it is just a compilation of simple parts or a chain of simple common sense steps. It is simply a matter of being calm, disciplined, and focused enough to let your brain see the parts.

The Nineteen Rules

Rules of Inference

Modus Ponens (M.P.)

$p \supset q$
 $p \therefore q$

Modus Tollens (M.T)

$p \supset q$
 $\sim q \therefore \sim p$

Hypothetical Syllogism (H.S.)

$p \supset q$
 $q \supset r \therefore p \supset r$

Disjunctive Syllogism (D.S.)

$p \vee q$
 $\sim p \therefore q$

Constructive Dilemma (C.D.)

$(p \supset q) \bullet (r \supset s)$
 $p \vee r \therefore q \vee s$

Absorption (Abs.)

$p \supset q$
 $\therefore p \supset (p \bullet q)$

Simplification (Simp.)

$p \bullet q$
 $\therefore p$

Conjunction (Conj.)

p
 $q \therefore p \bullet q$

Addition (Add.)

p
 $\therefore p \vee q$

Rules of Replacement

De Morgan's Theorems:
 (De M.)

$\sim(p \bullet q) \equiv (\sim p \vee \sim q)$
 $\sim(p \vee q) \equiv (\sim p \bullet \sim q)$

Commutation:
 (Com.)

$(p \vee q) \equiv (q \vee p)$
 $(p \bullet q) \equiv (q \bullet p)$

Association:
 (Assoc.)

$[p \vee (q \vee r)] \equiv [(p \vee q) \vee r]$
 $[p \bullet (q \bullet r)] \equiv [(p \bullet q) \bullet r]$

Distribution:
 (Dist.)

$[p \bullet (q \vee r)] \equiv [(p \bullet q) \vee (p \bullet r)]$
 $[p \vee (q \bullet r)] \equiv [(p \vee q) \bullet (p \vee r)]$

Double Negation (D.N.):

$p \equiv \sim\sim p$

Transposition (Trans.):

$(p \supset q) \equiv (\sim q \supset \sim p)$

Implication: (Impl.)

$(\sim p \supset q) \equiv (p \vee q)$

Equivalence: (Equiv.)

$(p \equiv q) \equiv [(p \supset q) \bullet (q \supset p)]$
 $(p \equiv q) \equiv [(p \bullet q) \vee (\sim p \bullet \sim q)]$

Exportation (Exp.)

$[(p \bullet q) \supset r] \equiv [p \supset (q \supset r)]$

Repetition (Rep.)

$p \equiv (p \vee p)$
 $p \equiv (p \bullet p)$

Application Practice

As you recall the first step of learning the nine rules involved just recognizing when a rule applied. Before you try some first step exercises with these new rules, let's look at some strategies and examples of pattern recognition with these new rules. Suppose we had the argument,

$$(A \supset B) \bullet (C \supset D) \\ \therefore (A \supset B) \bullet (\sim D \supset \sim C)$$

Your task is to identify which one of the new rules justifies this inference as valid. To focus, first ask, "**Has the whole line been changed in the conclusion or only a part?**" In this case only part of the premise has been changed in the conclusion. The right hand side of the conjunction ($\bullet$), $(C \supset D)$, has been replaced with $(\sim D \supset \sim C)$. Essentially an if-then statement ($\supset$) has been turned around and negations added to both the antecedent and the consequent. Scanning the new rules of replacement we see that the transposition rule, $(p \supset q) \equiv (\sim q \supset \sim p)$, justifies this move. Because $(p \supset q) \equiv (\sim q \supset \sim p)$ is a logical equivalence, $(C \supset D)$ can be replaced with $(\sim D \supset \sim C)$. So, the answer to the above problem would be "Trans."

Let's try another one, this time a whole line manipulation. Suppose we had,

$$(A \supset B) \bullet (B \supset A) \\ \therefore A \equiv B$$

Clearly in this case the whole line has been changed, because the major connective of the premise is a conjunction ($\bullet$) and the major connective of the conclusion is a biconditional ($\equiv$). A change in the major connective tells us that the whole line has been changed. In such situations we must scan the new rules, looking for a rule that has a conjunction on one side of the logical equivalence and a biconditional on the other, a rule with $(\bullet) \equiv (\equiv)$. Only the equivalence rule meets this condition, but before we conclude that this is the answer we must be sure the entire pattern fits. The first version of equivalence, $(p \equiv q) \equiv [(p \supset q) \bullet (q \supset p)]$, justifies the changing of two conditional statements connected by a conjunction to a biconditional, provided the two conditional statements have the special relationship, $(p \supset q), (q \supset p)$. The above premise does have two conditional statements in this special relationship, $(A \supset B), (B \supset A)$, so "Equiv." is the answer to this problem.

A whole line can also be manipulated where the major connective stays the same. In such cases, both sides of the major connective might be changed. For instance,

$$\sim(A \bullet B) \supset (D \bullet F) \\ \therefore \sim(D \bullet F) \supset \sim(A \bullet B)$$

has the same major connective in the premise and the conclusion, but the antecedent and the consequent have been reversed and a negation added to both sides. Notice that a negation has been added to $\sim(A \bullet B)$ to get $\sim \sim(A \bullet B)$, and a negation added to $(D \bullet F)$ to get $\sim(D \bullet F)$. Again the transposition rule, $(p \supset q) \equiv (\sim q \supset \sim p)$, allows for this manipulation.

$$p = \sim(A \bullet B)$$

$$q = (D \bullet F)$$

In applying these new rules you must be capable of the same level of abstraction as with the nine rules. As you know by now the following are both examples of modus ponens:

$$\begin{array}{ll} A \supset B & [(A \vee B) \supset (D \bullet F)] \supset (X \vee \sim P) \\ A & \therefore B \quad (A \vee B) \supset (D \bullet F) \therefore (X \vee \sim P) \end{array}$$

Similarly, both of the following are examples of transposition:

$$\begin{array}{ll} A \supset B & [(A \vee B) \supset (D \bullet F)] \supset (X \vee \sim P) \\ \therefore \sim B \supset \sim A & \therefore \sim(X \vee \sim P) \supset \sim[(A \vee B) \supset (D \bullet F)] \end{array}$$



Step 4: Rules of Replacement Exercises

For each of the following arguments state the rule of replacement that is being used. Indicate with an **X** those problems that do not have a rule of replacement justification.

1. $\sim A \supset B$
 $\therefore A \vee B$
2. $(C \supset D) \vee \sim(X \bullet Y)$
 $\therefore (C \supset D) \vee (\sim X \vee \sim Y)$
3. $(J \bullet K) \supset (D \supset I)$
 $\therefore J \supset [K \supset (D \supset I)]$
4. * $D \supset B$
 $\therefore \sim D \supset \sim B$
5. $O \supset [(X \supset Y) \bullet (Y \supset X)]$
 $\therefore O \supset (X \equiv Y)$
6. $\sim(R \vee S) \supset (\sim R \vee \sim S)$
 $\therefore \sim(S \vee R) \supset (\sim R \vee \sim S)$
7. $\sim(D \vee Y)$
 $\therefore \sim D \bullet \sim Y$
8. $(X \vee Y) \bullet (\sim X \vee \sim Y)$
 $\therefore [(X \vee Y) \bullet \sim X] \vee [(X \vee Y) \bullet \sim Y]$
9. * $(A \vee B) \supset (X \supset Y)$
 $\therefore B \vee A$
10. $[C \bullet (D \bullet E)] \vee (C \vee E)$
 $\therefore [(C \bullet D) \bullet E] \vee (C \vee E)$

11. $\sim(A \bullet D) \supset \sim X$
 $\therefore (A \bullet D) \supset \sim X$
12. $(H \supset \sim I) \supset (\sim I \supset \sim J)$
 $\therefore (H \supset \sim I) \supset (J \supset I)$
13. $(\sim K \vee L) \supset (\sim M \vee \sim M)$
 $\therefore (\sim K \vee L) \supset \sim M$
14. $[(\sim O \vee P) \vee \sim Q] \vee \sim(P \vee \sim Q)$
 $\therefore [\sim O \vee (P \vee \sim Q)] \vee \sim(P \vee \sim Q)$
15. $(R \vee \sim S) \vee (T \supset \sim P)$
 $\therefore (T \supset \sim P) \vee (R \vee \sim S)$
16. $[H \bullet (I \vee J)] \vee [H \bullet (K \supset \sim L)]$
 $\therefore H \bullet [(I \vee J) \vee (K \supset \sim L)]$
17. $[(\sim A \bullet B) \bullet (C \vee D)] \vee [\sim(\sim A \bullet B) \bullet \sim(C \vee D)]$
 $\therefore (\sim A \bullet B) \equiv (C \vee D)$
- 18.* $\sim A \supset B$
 $\therefore \sim A \vee B$
19. $\sim C \supset (D \bullet X)$
 $\therefore \sim(D \bullet X) \supset \sim \sim C$
20. $(\sim M \vee \sim \sim N) \bullet (O \equiv \sim \sim P)$
 $\therefore (\sim M \vee N) \bullet (O \equiv \sim \sim P)$
- 21.* $\sim(\sim A \bullet X)$
 $\therefore (A \bullet X)$
22. $A \supset (B \bullet C)$
 $\therefore A \supset (B \supset C)$
23. $\sim[A \supset (B \supset C)]$
 $\therefore \sim[(A \bullet B) \supset C]$
24. $\sim(\sim A \vee \sim B)$
 $\therefore \sim \sim(A \bullet B)$
25. $A \supset (B \bullet C)$
 $\therefore B$

Common Sense Origins

Before we try some second step exercises with these new rules, let's attempt to deflate some of the sensory overload you may be feeling by looking at some of these rules and their common sense origins.

De Morgan's Theorem

$$\begin{aligned}\sim(p \bullet q) &\equiv (\sim p \vee \sim q) \\ \sim(p \vee q) &\equiv (\sim p \bullet \sim q)\end{aligned}$$

We have already seen in Chapter 7 that a little reflection shows that the statement, "Alice and Barbara are not both going to the party," $\sim(A \bullet B)$, is the same as saying that "Either Alice is not going or Barbara is not going to the party," $(\sim A \vee \sim B)$. And, asserting a *neither-nor* is the same as stating a *both-not*, as in "Neither Alice nor Barbara are going to the party," $\sim(A \vee B)$, and "Alice and Barbara are both not going to the party," $(\sim A \bullet \sim B)$. So, there are two versions of De Morgan's theorem. Remember also that a *not-both* statement is not the same as a *both-not* statement, and a *neither-nor* statement is not the same as an *either not or not* statement. Thus, these versions must always be applied separately. A major mistake that students often make is to conclude $(\sim p \bullet \sim q)$, a part of the second version, from $\sim(p \bullet q)$, a part of the first version. If we knew that Alice and Barbara were not both coming to the party, we could not be certain from this

alone that Alice was not coming for sure and that Barbara was not coming for sure. Similarly, it would be a mistake to infer from, "Either John did not pass the final or he did not pass the course," ($\sim F \vee \sim C$), that "John passed neither the final exam nor the course," $\sim(F \vee C)$. As shown in Chapter 7,

$$\text{not both} = \text{either not...or not...} \quad = \quad \sim(_ \bullet _) = (\sim_ \vee \sim_)$$

$$\text{both not} = \text{neither... nor...} \quad = \quad \sim_ \bullet \sim_ = \sim(_ \vee _)$$

BUT

$$\text{not both} \neq \text{both not} \\ \sim(_ \bullet _) \neq (\sim_ \bullet \sim_)$$

$$\text{neither... nor...} \neq \text{either not...or not...} \\ \sim(_ \vee _) \neq (\sim_ \vee \sim_)$$

Commutation:

$$(p \vee q) \equiv (q \vee p) \\ (p \bullet q) \equiv (q \bullet p)$$

This rule simply reflects the common sense notion that "Either Alice or Barbara is going to the party," ($A \vee B$), is the same as "Either Barbara or Alice is going to the party," ($B \vee A$), and "Alice and Barbara are going to the party," ($A \bullet B$), is the same as "Barbara and Alice are going to the party," ($B \bullet A$).² Note, however, that this rule applies only to statements with the connectives *or* and *and*; it allows only statements with these connectives to be "turned around" or commuted in this manner. It is invalid to commute conditionals: ($p \supset q$) is not logically equivalent to ($q \supset p$). Given the premise, "If the police department does a good job, then crime will decrease," ($P \supset D$), it does not follow that "If crime decreases, then the police department must have done a good job," ($D \supset P$).³

² In grammar school we learned a similar commutation rule in mathematics that applied to addition and multiplication: $(3 + 2) = (2 + 3)$ and $(3 \times 2) = (2 \times 3)$. And then in algebra these common sense notions were generalized to " $a+b = b+a$ " and " $ab = ba$."

³ However, a biconditional statement can be commuted, $p \equiv q$ is logically equivalent to $q \equiv p$. The statement, "You can go out tonight if and only if you clean your room," ($O \equiv R$) is logically equivalent to "Cleaning your room is a necessary and sufficient condition for going out tonight," ($R \equiv O$). This will not be one of our rules though; we will prove this equivalence as a formal proof later.

Association:

$$[p \vee (q \vee r)] \equiv [(p \vee q) \vee r]$$

$$[p \bullet (q \bullet r)] \equiv [(p \bullet q) \bullet r]$$

This rule reflects the common sense notion that "Either Alice, or Barbara or Carol are going to the party," $[A \vee (B \vee C)]$, is the same as "Alice or Barbara are going to the party, or Carol is going to the party," $[(A \vee B) \vee C]$. And, "Alice is going to the party, and Barbara and Carol are also going to the party," $[A \bullet (B \bullet C)]$ is the same as "Alice and Barbara are going to the party, and Carol is going as well," $[(A \bullet B) \bullet C]$. Essentially the association rule states that when the logical connectives within a statement are all conjunctions ($\bullet$) or disjunctions ($\vee$), it does not matter where the parentheses go. They can be manipulated at will.⁴

However, note that association does not work with conditionals: $[p \supset (q \supset r)]$ is not logically equivalent to $[(p \supset q) \supset r]$. The statement, "If Weng has a quiz average of over 90%, then if he also gets at least 88% on the final exam then he will receive an A for the course," $[Q \supset (F \supset A)]$ is not saying the same thing as, "If Weng has a quiz average of over 90% only if he gets at least 88% on the final exam, then he will receive an A for the course," $[(Q \supset F) \supset A]$. The first statement refers to two conditions that together are sufficient for an A in the course, a high quiz average and a relatively high grade on the final exam, a likely state of affairs in many college courses; whereas, the second statement makes the absurd claim that a sufficient condition for an A in the course is that doing well on the final, which normally happens after the taking of quizzes, is a necessary condition for doing well on the quizzes!? In other words, it would be necessary to get at least 88% on the final exam before averaging over 90% on his quizzes, and this strange set of circumstances would then be sufficient for receiving an A in the course. So, unlike the association rule for conjunctions and disjunctions the parentheses cannot be moved at will for conditionals.⁵

Distribution:

$$[p \bullet (q \vee r)] \equiv [(p \bullet q) \vee (p \bullet r)]$$

$$[p \vee (q \bullet r)] \equiv [(p \vee q) \bullet (p \vee r)]$$

For most students the distribution rule is the most difficult to apply. But the common sense origin of its two versions is not difficult to see. The statement, "Alice is going to the party, and Barbara or Carol are going," $[A \bullet (B \vee C)]$, is the same as, "Either Alice and Barbara are going to the party or Alice and Carol are going to the party," $[(A \bullet B) \vee (A \bullet C)]$. And, the statement, "Either Rios passes the final exam or he will not receive the scholarship and not be eligible for

⁴ Similarly, in mathematics we learn that $(3 + 2) + 8$ is the same as $3 + (2 + 8)$, and $3 \times (5 \times 2)$ is the same as $(3 \times 5) \times 2$, and hence, $a + (b + c) = (a + b) + c$, and $a(bc) = (ab)c$.

⁵ We will see, however, that association is valid for biconditionals: $[(p \equiv q) \equiv r]$ is logically equivalent to $[p \equiv (q \equiv r)]$. Rather than have this as a rule in our system, we will save this for a very entertaining proof.

the football team," $[R \vee (\sim S \bullet \sim T)]$ is the same as, "Rios passes the final or he will not receive the scholarship, and Rios passes the final or he will not be eligible for the football team," $[(R \vee \sim S) \bullet (R \vee \sim T)]$.⁶

The important perspective to see for application purposes is that distribution always mixes conjunctions and disjunctions. It either "distributes" a conjunction through a disjunction (1st version), so that what was a conjunction ($\bullet$) becomes a disjunction ($\vee$), or it distributes a disjunction through a conjunction (2nd version), so that what was a disjunction ($\vee$) becomes a conjunction ($\bullet$). In trying to spot this rule in such difficult applications such as number 8 in the above exercises, remember that it is the only rule that changes ($\bullet$) to ($\vee$), or ($\vee$) to ($\bullet$).

Double Negation: $p \equiv \sim\sim p$

When we say that something is not impossible we mean that it is possible. When a bureaucrat says that an action was not unauthorized we know that it was authorized. Or, when in 1986, during the Iran-Contra scandal, Oliver North said that he did not believe that he went before Congress not believing that he was going to tell the truth, we know, after shaking our head a lot, that North was claiming that he went before Congress believing that he was going to tell the truth. In understanding these transformations we are applying the double negation rule, i.e. two negations cancel to make a positive statement. However, as intuitively obvious as this rule is, students often do not realize the full flexibility and power of its possible applications in terms of generating steps in a proof. Here is a trick question: "How many times can double negation be applied to, $A \supset (B \supset C)$?" Most students will answer 3 times. Some, a little more aware of the full meaning of a variable will reply 4 times, and a very few even more aware will reply 5 times. The answer, however, is an infinite number of times! To understand how, consider that applying double negation to $\sim\sim A$ could result in either just A or $\sim\sim\sim A$.

The important point to remember is that the double negation rule says that we can either put two negations on a statement or take two negations off a statement. In formal proofs some students will mistakenly think that we can conclude $\sim A$ from $\sim\sim A$, or that $\sim A \supset \sim B$ follows from $A \supset B$. The first application removes only one negation and hence would introduce a contradiction into a formal proof. The second application assumes that as long as the negations add up to two, we can take them from wherever we want! Both applications are invalid. Also, number 21 in the above exercises is not DN. DN cannot be applied to $\sim(\sim A \bullet X)$. The negation on the outside of the parentheses applies to the whole statement inside the parentheses, but the negation in front of the A applies only to the A .

Transposition: $(p \supset q) \equiv (\sim q \supset \sim p)$

Clearly from the statement, "If John passed the final exam, then he passed the course," ($F \supset C$),

⁶ In algebra we learn: $a(b + c) = ab + ac$ [Eg. $6(4 + 3) = (6 \times 4) + (6 \times 3)$]

we would be able to infer, "If John did not pass the course, then he did not pass the final exam," ($\sim C \supset \sim F$), and vice versa. If a sufficient condition for passing the course is to pass the final exam, then if the course was not passed we know that the sufficient condition was not fulfilled. However, as noted above it is a mistake to infer ($\sim p \supset \sim q$) from ($p \supset q$). Although a necessary condition for being pregnant is to be female, ($P \supset F$), it would surely not follow from this that if a person is not pregnant that person is not female, ($\sim P \supset \sim F$). The bottom line in applying transposition is to always remember that the consequent and the antecedent must be switched when negations are added to both. Also remember that rules of replacement can be applied in either direction of the ($\equiv$) symbol. So given ($\sim A \supset \sim B$), we could infer either ($B \supset A$) or ($\sim \sim B \supset \sim \sim A$). The latter application might be useless, silly, or at least inelegant,⁷ but it would be valid.

Implication: $(\sim p \supset q) \equiv (p \vee q)$

If either Alice or Barbara are going to the party, ($A \vee B$), then if Alice is not going, then Barbara will go, ($\sim A \supset B$). And, as noted at the beginning of this chapter an inference such as,

Either the Start II treaty is ratified or this landmark treaty will not be worth the paper it is written on. Therefore, if the Start II treaty is not ratified, this landmark treaty will not be worth the paper it is written on.

$$\begin{array}{l} R \vee \sim W \\ \therefore \sim R \supset \sim W \end{array}$$

is valid. It would also follow from, "If Leina does not pass the final, she does not pass the course," ($\sim F \supset \sim C$), that, "Either Leina passes the final or she does not pass the course," ($F \vee \sim C$).

The important application point to remember about implication is that this rule allows us to move back and forth between disjunctions ($\vee$) and conditionals ($\supset$), provided that a negation is either added (to the antecedent in moving to a conditional) or subtracted (from the antecedent in moving to a disjunction) in making the transformation.

Equivalence: $(p \equiv q) \equiv [(p \supset q) \bullet (q \supset p)]$
 $(p \equiv q) \equiv [(p \bullet q) \vee (\sim p \bullet \sim q)]$

As you recall, in Chapter 7 we discussed the meaning of statements like, "You go out tonight with your friends if and only if you clean your room," ($O \equiv R$). There we noted that the use of *if*

⁷ It would be inelegant because with two applications of double negation, $\sim \sim B \supset \sim \sim A$ would become $B \supset A$. So why conclude something in three steps (transposition and two double negations), when we can infer the same thing in one step?

and only if is a way of specifying a necessary and sufficient condition, $[(O \supset C) \bullet (C \supset O)]$. We can now add that this statement is also equivalent to, "Either you go out and clean your room, or you don't go out and don't clean your room," $[(O \bullet C) \vee (\sim O \bullet \sim C)]$.

Note that the important application point for equivalence is that although all the rules make use of the biconditional symbol ($\equiv$), only equivalence has an application result that produces this connective, $(p \equiv q)$. In looking at 17 in the above step 4 exercises the only possible replacement rule would be equivalence, because no other rule allows us to transform a premise into a biconditional statement. Noting this, your original hypothesis should be that 17 is equivalence, and then you would focus on whether all the parts fit.

It is also worth noting at this point that because there are two versions of equivalence, and both versions have $p \equiv q$ as a possible result, unlike a rule such as De Morgan where it is invalid to conclude from a part of one version a part of the other version—we can't conclude $\sim p \bullet \sim q$ from $\sim(p \bullet q)$ —with equivalence it is possible to move validly from $[(p \supset q) \bullet (q \supset p)]$ to $[(p \bullet q) \vee (\sim p \bullet \sim q)]$, as we will see. But it takes two steps. Can you see the steps?

Exportation: $[(p \bullet q) \supset r] \equiv [p \supset (q \supset r)]$

The intuitive validity of exportation is seen by noting the logical equivalence of statements like, "If we make the car payment and also pay our medical bills this month, then we will not have enough money for basic necessities," $[(C \bullet M) \supset \sim B]$, and "If we make the car payment, then if we also pay our medical bills we will not have enough money for basic necessities," $[C \supset (M \supset \sim B)]$.

An important application point to note is that exportation cannot be applied to any statement with the form $[p \supset (q \bullet r)]$. A common mistake made by students in formal proofs is to conclude statements with the form $[p \supset (q \supset r)]$ or $[(p \supset q) \supset r]$ from $[p \supset (q \bullet r)]$. (Note number 22 in the exercises above.⁸) But the statement (1), "If we make the car payment this month, then we will also pay our medical bills and not have enough money for basic necessities," $[C \supset (M \bullet \sim B)]$, clearly has a different meaning than (2), "If we make the car payment, then if we also pay our medical bills we will not have enough money for basic necessities," $[C \supset (M \supset \sim B)]$, and (3), "If we make the car payment only if we also pay our medical bills, then we will not have enough money for basic necessities," $[(C \supset M) \supset \sim B]$. Statement (1) asserts that making the car payment is a sufficient condition for also paying the medical bills, and statement (3) has as a sufficient condition for not having enough money for basic necessities the claim that a necessary condition for making the car payment is paying the medical bills! Whereas, all that is intended by (2) is that if both the car payment and the medical bills are paid, then there will not be enough money for basic necessities, $[(C \bullet M) \supset \sim B] \equiv [C \supset (M \supset \sim B)]$.

⁸ But also note that number 23 in the above exercises is a correct exportation. As a rule of replacement, exportation can be applied to part of a line, in this case the part inside the $\sim[]$.

Repetition: $p \equiv (p \vee p)$
 $p \equiv (p \bullet p)$

Our last logical equivalence is so obvious it may seem silly even to have to state it. Surely, if we know that John passed the final exam, (F), then we know that he either passed the final exam or he passed the final exam, $(F \vee F)$, and he passed the final exam and he passed the final exam, $(F \bullet F)$. However, this rule is not only needed to make our rules complete—to have a rule-system that will enable us to construct a formal proof of validity for any argument that can be translated into propositional logic—but also by comparison with $p \equiv (p \supset p)$ and $p \equiv (p \equiv p)$, which are both **invalid**, we see that we must be careful with even the simplest of inferences.

Strategies for Pattern Recognition Revisited

Recall the major point behind all logical analysis: We get straight on the simplest components of reasoning and then use these mini-steps to analyze or produce much more complicated reasoning trails and claims. With this little excursion into the common sense background to the rules of replacement now complete, we are ready to continue the steps necessary for constructing your own formal proofs using all the rules. Aspiring painters must first learn to mimic the styles of the great masters. As in Chapter 9, Step 2, it is time for you to attempt to mimic completed reasoning trails by figuring out how the trails were constructed. But let's do one problem together first.

As in Chapter 9, Step 2, the following Step 5 exercises require that you justify each line by citing the premise or premises and the rule used. Any one of the nineteen rules may now be involved. At first this will involve a lot of staring and possible sensory overload. Recall also that what made life bearable in the Step 2 exercises was two strategies that allowed us to focus and test specific rule application possibilities, rather than just randomly stare at the rules. These same strategies will of course continue to work with lines that involve application of the first nine rules. We can now add, however, a third strategy that we were unconsciously using in doing Step 4 exercises. Because any application of the ten rules of replacement will necessarily involve a modification of a premise—either part of a line will be changed or a whole line—we can use this to our advantage as a strategy: We can examine whether the line we need to justify seems to be a modification of any previous line. For instance, consider the following proof:

1. $A \supset \sim(B \vee C)$
2. $(\sim B \bullet \sim C) \supset D$
3. $D \supset A$
4. $\underline{(A \equiv D) \supset X \quad \therefore X}$
5. $A \supset (\sim B \bullet \sim C)$
6. $A \supset D$
7. $(A \supset D) \bullet (D \supset A)$
8. $A \equiv D$

9. X

The first line to be justified is line 5. In comparing line 5 with the premises we see that it is very similar to line 1. The same statements are involved (A's, B's, and C's), and these lines have the same antecedent (A) and the same conditional ($\supset$) major connective. The only difference is that the consequent in line 1 is $\sim(B \vee C)$ and the consequent in line 5 is $(\sim B \bullet \sim C)$. In looking at the rules of replacement we know that one of the versions of the De Morgan rule, $\sim(p \vee q) \equiv (\sim p \bullet \sim q)$, justifies this modification.

$$\sim(B \vee C) \equiv (\sim B \bullet \sim C) \text{ De M.}$$

So, we would now have:

1. $A \supset \sim(B \vee C)$
2. $(\sim B \bullet \sim C) \supset D$
3. $D \supset A$
4. $\underline{(A \equiv D) \supset X} \quad \therefore X$
5. $A \supset (\sim B \bullet \sim C) \quad (1) \text{ De M.}$
6. $A \supset D$
7. $(A \supset D) \bullet (D \supset A)$
8. $A \equiv D$
9. X

This strategy of looking for a modification of a line does not appear to work for line 6. Although line 3 has the same letters as 6, looking at the rules of replacement we see that there is no rule that allows us to turn a conditional statement around. Remember that commutation does not apply to ($\supset$).

$$(D \supset A) \neq (A \supset D)$$

The closest we have to this would be transposition, but this rule requires negations to be added to both the antecedent and the consequent. The first strategy used in Step 2 exercises of seeing whether line 6 is a part of a previous line also fails. Hence, we are back to the strategy of focusing on the connective, and in this case then examining all the rules with ($\supset$) as the major connective of the conclusion.

Since we have already ruled out all the rules of replacement, we focus on only HS and Absorption. Absorption is also eliminated because the consequent of the conclusion always involves two statement parts and line 6 only has one.

Thus, we are left with hypothetical syllogism as the only viable hypothesis. This then means we would be looking for either,

1. $A \supset \sim(B \vee C)$
 ? $\sim(B \vee C) \supset D \therefore A \supset D$

or

5. $A \supset (\sim B \bullet \sim C)$
 ? $(\sim B \bullet \sim C) \supset D \therefore A \supset D$

since both would produce the desired conclusion by hypothetical syllogism. Because we don't have $\sim(B \vee C) \supset D$ as a premise, and do have $(\sim B \bullet \sim C) \supset D$ in line 2, the second alternative works and we have found:

(5) $A \supset (\sim B \bullet \sim C)$
 (2) $(\sim B \bullet \sim C) \supset D \therefore (6) A \supset D$ HS

$p \supset q$
 $q \supset r \therefore p \supset r$

We now have:

1. $A \supset \sim(B \vee C)$
 2. $(\sim B \bullet \sim C) \supset D$
 3. $D \supset A$
 4. $(A \equiv D) \supset X \therefore X$
 5. $A \supset (\sim B \bullet \sim C)$ (1) De M.
 6. $A \supset D$ (5)(2) HS
 7. $(A \supset D) \bullet (D \supset A)$
 8. $A \equiv D$
 9. X

Next line 7. Line 7 is definitely not a part of a previous line, nor does it appear to be a modification of a previous line. So let's try focusing on the connective. As a conjunction ($\bullet$) for the major connective, potentially we have the conjunction rule for the first nine rules. Again, we don't need to worry about any of the rules of replacement, because we have eliminated all of them when we said that it did not look like line 7 was a modification of any line above. In focusing on conjunction, you should see,

(6) $A \supset D$ p
 (3) $D \supset A \therefore (A \supset D) \bullet (D \supset A)$ q $\therefore p \bullet q$

So, we now have:

1. $A \supset \sim(B \vee C)$
 2. $(\sim B \bullet \sim C) \supset D$
 3. $D \supset A$
 4. $(A \equiv D) \supset X \therefore X$
 5. $A \supset (\sim B \bullet \sim C)$ (1) De M.
 6. $A \supset D$ (5)(2) HS
 7. $(A \supset D) \bullet (D \supset A)$ (6)(3) Conj.
 8. $A \equiv D$

9. X

Note that line 7 could have followed from either $(D \supset A) \bullet (A \supset D)$ by commutation or $A \equiv D$ by equivalence. However, neither of these lines exist above line 7. The statement $A \equiv D$ exists in line 8, but a line below the line we are on can never be used to justify the line we are on. However, because there is the relationship of equivalence between line 7 and 8, we have our answer for line 8, because:

$$[(A \supset D) \bullet (D \supset A)] \equiv (A \equiv D)$$

So we now have:

- | | |
|--|--------------|
| 1. $A \supset \sim(B \vee C)$ | |
| 2. $(\sim B \bullet \sim C) \supset D$ | |
| 3. $D \supset A$ | |
| 4. <u>$(A \equiv D) \supset X \quad \therefore X$</u> | |
| 5. $A \supset (\sim B \bullet \sim C)$ | (1) De M. |
| 6. $A \supset D$ | (5)(2) HS |
| 7. $(A \supset D) \bullet (D \supset A)$ | (6)(3) Conj. |
| 8. $A \equiv D$ | (7) Equiv. |
| 9. X | |

In working on line 7, sometimes students will be trapped into focusing on line 4, because $A \equiv D$ is part of line 4. By now, however, if you know the first nine rules well, you know that no rule of inference works like this:

$$(A \equiv D) \supset X$$
$$(??) \therefore A \equiv D \quad \text{X}$$

Plus, no rule of replacement works like this:

$$(A \equiv D) \supset X \therefore (A \supset D) \bullet (D \supset A) \quad \text{X}$$

Sometimes you will just see the justification without going through any strategies consciously. Remember, that often the line above is used to justify the line you are on, and by looking at line 8 you may see a connection with the rule of equivalence immediately. After lots of practice, many students will say that the connections just start to “pop out.”

Finally, for line 9, because it does not have any connectives, it is most likely that strategy 1 will work, that the X is concluded as part of a line above and modus ponens, disjunctive syllogism, or simplification is involved.⁹ Viewed from this perspective, we see that line 4 has the X in the

⁹ In such situations the only other possibilities would be double negation, $X \equiv \sim\sim X$, or repetition, $X \equiv (X \bullet X)$ or $X \equiv$

right position for a modus ponens and line 8 matches the antecedent of line 4 giving us:

(4) $(A \equiv D) \supset X$	$p \supset q$
(8) $A \equiv D \therefore X$	$p \therefore q$

So we now have the completed proof:

1. $A \supset \sim(B \vee C)$	
2. $(\sim B \bullet \sim C) \supset D$	
3. $D \supset A$	
4. <u>$(A \equiv D) \supset X \therefore X$</u>	
5. $A \supset (\sim B \bullet \sim C)$	(1) De M.
6. $A \supset D$	(5)(2) HS
7. $(A \supset D) \bullet (D \supset A)$	(6)(3) Conj.
8. $A \equiv D$	(7) Equiv.
9. X	(4)(8) MP

What emerges from this practice problem is that we can simply add the strategy for the rules of replacement to our previous two strategies for the rules of inference used in Chapter 9.

Strategy 1: See if the line that you are trying to justify is a part of a premise, and then try to match with modus ponens, disjunctive syllogism, or simplification.

Strategy 2: Focus on the connective of the conclusion, and then try to match with an appropriate rule of inference. Remember: If the major connective of the conclusion is a negation ($\sim$), try to match with modus tollens; if the major connective of the conclusion is a conditional ($\supset$), try to match with either absorption or hypothetical syllogism; if the major connective in the conclusion is a disjunction ($\vee$), try to match with addition or constructive dilemma; and, if the major connective is a conjunction, try to match with the conjunction rule ($\bullet$).

Strategy 3: See if the line that you are trying to justify is a modification of a previous line, and then try to match with a rule of replacement.

The important point to remember in attempting the following exercises is that provided that you have mastered Steps 1 and 4,¹⁰ these strategies will give you a way of staying focused, disciplined, and calm in developing hypotheses on possible rule applications, and testing each one in a trial and error process. If you find yourself so overwhelmed that all you are doing is staring randomly at the list of rules, you will create a psychological slippery slope process of

($X \vee X$). But we don't have $\sim\sim X$, or ($X \bullet X$) or ($X \vee X$) in any of the lines above.

¹⁰ If you have not mastered Steps 1 and 4, you will be wasting your time trying to do the Step 5 exercises. Go back and do the Step 1 and 4 exercises again. Often in life we have to go backwards before we can go forward.

anxiety and sensory overload, and you will not give your brain a chance to see the simplest rule applications.



Step 5 Exercises

State the justification for each line that is not a premise. There are no incorrect lines and any one of the nineteen rules could be involved.

#1

1. $A \supset B$
2. $C \supset \sim B \quad \therefore A \supset \sim C$
3. $\sim\sim B \supset \sim C$
4. $B \supset \sim C$
5. $A \supset \sim C$

#2

1. $(D \bullet E) \supset F$
2. $(D \supset F) \supset (\sim G \vee Y)$
3. $E \quad \therefore \sim G \vee Y$
4. $(E \bullet D) \supset F$
5. $E \supset (D \supset F)$
6. $D \supset F$
7. $\sim G \vee Y$

#3*

1. $(A \vee T) \supset [J \bullet (K \bullet L)]$
2. $T \quad \therefore J \bullet K$
3. $T \vee A$
4. $A \vee T$
5. $J \bullet (K \bullet L)$
6. $(J \bullet K) \bullet L$
7. $J \bullet K$

#4

1. $(M \vee N) \supset (C \bullet D)$
2. $\sim C \quad \therefore \sim M$
3. $\sim C \vee \sim D$
4. $\sim(C \bullet D)$
5. $\sim(M \vee N)$
6. $\sim M \bullet \sim N$
7. $\sim M$

#5

1. $(Q \supset Q) \supset \sim R$
2. $\sim Q \supset (R \bullet \sim Q) \quad \therefore \sim R$
3. $Q \vee (R \bullet \sim Q)$
4. $(Q \vee R) \bullet (Q \vee \sim Q)$
5. $(Q \vee \sim Q) \bullet (Q \vee R)$
6. $Q \vee \sim Q$
7. $\sim Q \supset \sim Q$
8. $Q \supset Q$
9. $\sim R$

#6

#7

1. $D \equiv Y$
2. $D \supset (\sim X \supset P)$
3. $\sim X$
4. $\sim P \quad \therefore \sim Y$
5. $(D \supset Y) \bullet (Y \supset D)$
6. $(Y \supset D) \bullet (D \supset Y)$
7. $Y \supset D$
8. $Y \supset (\sim X \supset P)$
9. $\sim X \bullet \sim P$
10. $\sim(X \vee P)$
11. $\sim(\sim X \supset P)$
12. $\sim Y$

1. $(D \bullet E) \supset F$
2. $\sim F \vee (G \bullet H)$
3. $D \equiv E \quad \therefore D \supset G$
4. $(D \supset E) \bullet (E \supset D)$
5. $D \supset E$
6. $D \supset (D \bullet E)$
7. $D \supset F$
8. $(\sim F \vee G) \bullet (\sim F \vee H)$
9. $\sim F \vee G$
10. $\sim \sim F \supset G$
11. $F \supset G$
12. $D \supset G$

#8

1. $A \supset Z$
2. $Z \supset [A \supset (R \vee S)]$
3. $R \equiv S$
4. $\sim(R \bullet S) \quad \therefore \sim A$
5. $A \supset [A \supset (R \vee S)]$
6. $(A \bullet A) \supset (R \vee S)$
7. $A \supset (R \vee S)$
8. $(R \bullet S) \vee (\sim R \bullet \sim S)$
9. $\sim R \bullet \sim S$
10. $\sim(R \vee S)$
11. $\sim A$

#9

1. $A \supset B$
2. $B \supset C$
3. $C \supset A$
4. $A \supset \sim C \quad \therefore \sim A \bullet \sim C$
5. $A \supset C$
6. $(A \supset C) \bullet (C \supset A)$
7. $A \equiv C$
8. $(A \bullet C) \vee (\sim A \bullet \sim C)$
9. $\sim \sim A \supset \sim C$
10. $\sim A \vee \sim C$
11. $\sim(A \bullet C)$
12. $\sim A \bullet \sim C$

#10

1. $T \bullet (U \vee V)$
2. $T \supset [U \supset (W \bullet X)]$
3. $(T \bullet V) \supset \sim(W \vee X) \quad \therefore W \equiv X$
4. $(T \bullet U) \supset (W \bullet X)$
5. $(T \bullet V) \supset (\sim W \bullet \sim X)$
6. $[(T \bullet U) \supset (W \bullet X)] \bullet [(T \bullet V) \supset (\sim W \bullet \sim X)]$
7. $(T \bullet U) \vee (T \bullet V)$
8. $(W \bullet X) \vee (\sim W \bullet \sim X)$
9. $W \equiv X$

Subroutines

As you applied the strategies to the above lines, it is normal to focus only line by line. As a preparation for our final step, it is worth taking some time to examine the strategies and blocks of steps in the above exercises. In the final step you will be asked to start with some premises and create a symbolic trail to the desired conclusion. With the full nineteen rules you now have an enormous flexibility in creating reasoning trails. As in life you have many, many paths to choose from. How do we make the right choice? How do you find the right reasoning trail when there are so many trails to follow?

Part of the secret in creating successful formal proofs is recognizing what we will call **subroutines** and being proficient with the rules such that you begin to see blocks of steps in looking at one or two premises. *In being able to see where a trail will go quickly by seeing blocks of steps, you can experiment faster.* For instance, consider the following situation.

1. $A \bullet B$
- 2.
3. $B \supset (D \bullet F)$
- 4.
5. $F \supset \sim X$

Someone proficient with the nineteen rules will be able to see ahead of actually constructing the proof that $\sim X$ will be available in this proof if needed. By the end of this chapter you will also be able to do this. The trick is being able to recognize particular repetitive steps, or subroutines, that are used often in different formal proofs. Because line 1 is a conjunction, the following subroutine is available.

1. $A \bullet B$
-
6. A (1) Simp.
7. $B \bullet A$ (1) Com.
8. B (7) Simp.

This subroutine shows that for future use whenever a conjunction ($\bullet$) is the major connective of a premise, we can now conclude both components of the conjunction and this is true regardless of the complexity of each component. (Notice that lines 6 and 7 of problem 6 in the Step 5 exercises involved this subroutine.)

Thus, in this problem we know that both A and B are available. But if B is available, then $D \bullet F$ is available because B will work with modus ponens and line 3. If $D \bullet F$ is available, then both D and F are available due to the applicability of the same subroutine that gave us A and B . If F is available, then $\sim X$ is available because of line 5 and MP. Hence, in doing this problem someone proficient in doing formal proofs might write down on scratch paper, A , B , D , F , and

$\sim X$ —indicating that they are available—before doing even a single step.

Another common subroutine was used in problem 1 of the above Step 5 exercises. Often you may be faced with,

- | | | |
|-----------------------|----|-----------------------|
| 1. $\sim A \supset X$ | or | 1. $X \supset \sim C$ |
| 2. $A \supset Y$ | or | 2. $Y \supset C$ |

where it appears that there is not much you can do to get lines 1 and 2 to work together. Often progress is made in proofs by using two premises together and only the first nine rules allow us to use two premises. For this reason, the second ten rules are often used to set up a situation where one of the nine rules will work. In the above situations, we have conditional statements ($\supset$) where either the antecedents or the consequents are opposite each other. As shown in problem 1 the trick here is to use transposition to set up hypothetical syllogism. So in the above situations we could do,

- | | | | |
|----------------------------|------------|----------------------------|------------|
| 1. $\sim A \supset X$ | | 1. $X \supset \sim C$ | |
| 2. $A \supset Y$ | | 2. $Y \supset C$ | |
| 3. $\sim Y \supset \sim A$ | (2) Trans. | 3. $\sim C \supset \sim Y$ | (2) Trans. |
| 4. $\sim Y \supset X$ | (3)(1) HS | 4. $X \supset \sim Y$ | (1)(3) HS |

Implementing such a subroutine does not guarantee progress or success, but for future use it is something to try whenever we find two conditional statements in a proof and either the antecedents or consequents are opposite each other.

Part of another helpful subroutine is shown in problem 2. In the future you may be faced with,

1. $A \supset (B \supset C)$
2. B

and because the first nine rules can be applied to only whole lines, the B in line 2 cannot be used to implement a modus ponens with line 1. That is, the B cannot be used with the consequent of line 1 to get C . However, by using exportation, commutation, and exportation again, the B in line 1 can be moved to work with line 2 as a modus ponens.

1. $A \supset (B \supset C)$
2. B
3. $(A \bullet B) \supset C$ (1) Exp.
4. $(B \bullet A) \supset C$ (3) Com.
5. $B \supset (A \supset C)$ (4) Exp.
6. $A \supset C$ (5)(2) MP

Often subroutines can be combined to obtain lots of steps. By combining the equivalence rule with the previous subroutine that used commutation and simplification, four steps can be generated given any biconditional statement as follows:

1. $A \equiv B$
2. $(A \supset B) \bullet (B \supset A)$ (1) Equiv.
3. $A \supset B$ (2) Simp.
4. $(B \supset A) \bullet (A \supset B)$ (2) Com.
5. $B \supset A$ (4) Simp.

De Morgan's theorem can also be used with commutation and simplification, as follows,

1. $\sim(A \vee B)$
2. $\sim A \bullet \sim B$ (1) De M.
3. $\sim A$ (2) Simp.
4. $\sim B \bullet \sim A$ (2) Com.
5. $\sim B$ (4) Simp.

For future use, when you see a negated disjunction in a proof it is automatic to do the above subroutine at least on scratch paper to see what pieces are available.

A fruitful subroutine that also combines commutation and simplification was used in lines 2-7, problem 5. Often you may be faced with this type of conditional statement:

1. $\sim A \supset (B \bullet C)$

Although we know from common sense that $\sim A \supset B$ and $\sim A \supset C$ individually follow from this statement, we cannot deduce either of these statements in one step by simplification.¹¹ (If Alice not going to the party causes both Barbara and Carol to go, then clearly if Alice does not go, Barbara will go, and if Alice does not go, then Carol will go.) Since we claim that our nineteen rules give us a complete system—that a formal proof exists for any valid argument that can be translated into propositional logic—some chain of reasoning must exist using our system of rules that will enable us to derive $\sim A \supset B$ and $\sim A \supset C$. Here's how:

1. $\sim A \supset (B \bullet C)$
2. $A \vee (B \bullet C)$ (1) Impl.
3. $(A \vee B) \bullet (A \vee C)$ (2) Dist.
4. $A \vee B$ (3) Simp.

¹¹ Remember that only when the major connective is a conjunction can simplification be used. The major connective in this line is a conditional.

- | | |
|------------------------------------|-----------|
| 5. $\sim A \supset B$ | (4) Impl. |
| 6. $(A \vee C) \bullet (A \vee B)$ | (3) Com. |
| 7. $A \vee C$ | (6) Simp. |
| 8. $\sim A \supset C$ | (7) Impl. |

This implication-distribution subroutine not only solves the problem of deriving $\sim A \supset B$ and $\sim A \supset C$ in our system, but shows the power and flexibility of the nineteen rules. From a single line, 7 steps are derived. For future reference this sequence of steps will always work for conditional statements ($\supset$) that have a conjunction ($\bullet$) as the connective for the consequent.

An elegant subroutine is shown in problem 4. Here is a similar series of steps:

- | | |
|------------------------------|-----------|
| 1. $A \supset (B \bullet C)$ | |
| 2. $\sim B$ | |
| 3. $\sim B \vee \sim C$ | (2) Add. |
| 4. $\sim(B \bullet C)$ | (3) De M. |
| 5. $\sim A$ | (1)(4) MT |

Note that with the addition of commutation, this same routine would work if line 2 were $\sim C$ instead of $\sim B$. We would simply add $\sim B$ to $\sim C$, then commute for $\sim B \vee \sim C$. This shows that whenever you have a conditional statement with a conjunction as the connective in the consequent (1), and also have a negation in another premise (2) of one of the components of the conjunction consequent, adding the negation of the other component (as in line 3) starts us on our way to get these lines to work together.

There are many more subroutines than this, and most you will have to discover for yourself as you become more experienced in attempting formal proofs. You will probably even create your own, because it is possible to adopt a particular style in doing proofs. What you need is practice and the challenge of facing nothing but empty space under the premises and the thought that that space can be filled with an infinite number of possible symbolic trails. Shortly you will be given considerable practice.

However, before we leave the subject of subroutines, note that due to the particularities of our system, often certain "bookkeeping" steps are needed to implement the rules properly. Problem 9 in the above Step 5 exercises (lines 4, 9 and 10) shows what is needed to implement implication when the antecedent of the conditional statement is not negated. Because our implication rule, $(\sim p \supset q) \equiv (p \vee q)$, states that the antecedent of the conditional must have a negation before we can move to a disjunction, when there is no negation we have to "trick" our system a little to implement this rule by using double negation. Here's how:

- | | |
|---------------------------------------|--------|
| 1. $A \supset (B \bullet C)$ | |
| 2. $\sim\sim A \supset (B \bullet C)$ | (1) DN |

$$3. \sim A \vee (B \bullet C) \quad (2) \text{ Impl.}$$

The implication rule directs us to remove one negation from the antecedent of the conditional, change the connective to a disjunction, and not change what was the consequent. So if there is no negation on the antecedent, we can't start the rule. But if there are two negations, we can implement the rule by removing one!

A similar trick can be used in situations such as the following:

- | | | |
|-------------------------|--|----------------------------|
| 1. | 2. | 3. |
| 1. $A \supset \sim Y$ | 1. $(B \bullet \sim C)$ | 1. $X \vee \sim Y$ |
| 2. $\underline{Y}$ | 2. $(\sim \sim B \bullet \sim C)$ (1) DN | 2. $\underline{Y}$ |
| 3. $\sim \sim Y$ (2) DN | 3. $\sim(\sim B \vee C)$ (2) De M. | 3. $\sim Y \vee X$ (1) Com |
| 4. $\sim A$ (1)(3) MT | | 4. $\sim \sim Y$ (2) DN |
| | | 5. X (3)(4) DS |

The first example (1) shows what we must do to implement modus tollens when originally we don't have a second premise that negates the consequent of a conditional statement. Technically, Y is not the negation of $\sim Y$, but since $\sim \sim Y$ is, we simply use double negation as a little bookkeeping subroutine. The second example (2) shows how to implement De Morgan's theorem when a conjunction does not have both components negated. Again double negation is used. One of the versions of the De Morgan rule directs us to remove a negation from each side of a conjunction, change the connective from a conjunction to a disjunction, and then place a negation around the entire disjunction. Following these steps will leave us with a $\sim B$ —we remove as directed one negation from $\sim \sim B$. The third example (3) shows how to implement the disjunction rule by first using commutation and double negation as bookkeeping steps.

As you become more proficient with the rules you may become frustrated with the time needed for bookkeeping steps. You will begin to have insights and be able to think much faster than you can write down each formal step. At that stage it is appropriate to use a short-cut technique of notation that combines bookkeeping steps with other rules. Here are some typical short-cut moves:

- | | |
|-----------------------------------|--------------------------------------|
| 1. | 2. |
| 1. $\underline{A \bullet B}$ | 1. $\sim X \vee Y$ |
| 2. B (1) Com. + Simp. | 2. $\underline{X}$ |
| | 3. Y (2)(1) DN + DS |
| 3. | 4. |
| 1. $\underline{D \supset Z}$ | 1. $\underline{\sim(B \vee \sim C)}$ |
| 2. $\sim D \vee Z$ (1) DN + Impl. | 2. $\sim B \bullet C$ (1) De M. + DN |

5.

1. $(A \bullet B) \bullet C$

2. A (1) Ass. + Simp.

6.

1. $P \supset \sim S$

2. $S \supset \sim P$ (1) Trans. + DN

It is important to note, however, that these examples are short-cut *notation* techniques. Steps are not skipped. In working with formal systems as with computers, every step must be rigorously specified.

Direction, Strategies, and Working Backwards

As in life, one of the ways to cut down the number of possible choices that confront us is to have a definite goal in mind. In constructing formal proofs our goal is to derive the conclusion. Most often in looking at the conclusion you will not see every step ahead of time, but a little common sense in comparing the premises with the conclusion will constrain what you should try first. For instance, consider the following problem.

1. $A \supset \sim B$
2. $C \supset B$
3. $\sim C \supset X \quad \therefore \quad \sim X \supset \sim A$

Because the premises are all conditionals and the conclusion is also a conditional, it would be wise to try the strategy that some combination of rules that involve conditionals ($\supset$) will work in solving this problem. So we might try using some combination of hypothetical syllogism and transposition.

Sometimes, however, the conclusion does not have a clear connection with the premises. In such cases a useful strategy is to work backwards on the conclusion using only the rules of replacement. Because implementing a rule of replacement produces a statement that is logically equivalent, using this strategy will reveal other possible goals. For instance, in the following problem,

1. $A \supset \sim B$
2. A
3. $H \supset \sim D$
4. D $\quad \therefore \quad \sim(B \vee H)$

it is difficult to see how we would arrive at a negated disjunction as a conclusion from the given premises. But by applying De Morgan's theorem to the conclusion on scratch paper, we see that a possible goal for this proof is to find a $\sim B$ and a $\sim H$ separately, then combine them by

conjunction—because $\sim(B \vee H)$ is logically equivalent to $\sim B \bullet \sim H$, and this statement can be produced by combining a $\sim B$ and a $\sim H$ by conjunction. If this were not done, we might be tempted to change by implication the conditional statements in lines 1 and 3 to disjunctions, because we see that the conclusion involves a disjunction. Although valid applications of a rule, this would take us on a long messy trail. After seeing what the conclusion is logically equivalent to, we see that it would be better to just do a modus ponens with lines 1 and 2, deriving $\sim B$, one half of what we are looking for. Often playing with the conclusion a little on scratch paper using rules of replacement produces alternative statements that we might then blunder across in going forward in the proof. A common strategy in difficult proofs is to work backwards a little, then blunder forward and hope that the two trails meet!

The following exercises are designed to eventually push you to a state of rigor that mimics the process of symbolic manipulation that is behind much of our technology. The first problems require only a few steps, but the last problem (after the translation exercises) that I have labeled Super Challenging requires approximately 50 steps! You will probably not be able to solve every problem.¹² However, your goal is not to solve every problem, but rather to experiment, try strategies, fail, start again, and above all, stay disciplined and create only valid trails. The biggest mistake you can make is to be impatient and derive the conclusion using a trail that contains misapplications of the rules. Here is a typical example of a student proof with incorrect lines marked with **xx**'s,

- | | |
|------------------------------|-------------------------------|
| 1. $(X \bullet Y) \supset Z$ | |
| 2. $\sim Z$ | |
| 3. $(T \vee P) \supset X$ | |
| 4. Y | $\therefore \sim T \supset P$ |
| 5. $\sim(X \bullet Y)$ | (1)(2) MT |
| 6. $\sim X$ | (5) Simp. xx |
| 7. $\sim T \vee P$ | (3)(6) MT xx |
| 8. $\sim \sim T \supset P$ | (7) Impl. |
| 9. $T \supset P$ | (8) DN |

This impatient proof has only five steps (5-9), but my proof for this problem has seven steps. The student who did this proof either did not check his or her steps or was so concerned with producing the conclusion that lines 6 and 7 were "hallucinated" as valid. Often I will give back a quiz like this and a fairly good student will immediately see the mistake or mistakes and usually remark, "Why didn't I see this?" Laziness, inattention to detail, and/or not understanding the rules are possible explanations. However, my theory is that in many cases the student did not see the mistakes because we are so much a part of a results-oriented society, so concerned with success and the competitive accomplishment of goals, that we have not been encouraged to stop

¹² As I write this, over the past twenty years, only thirteen students have been able to solve the Super Challenging problem, and ten of these students have been women! Much to the dismay of the macho men in my classes.

and enjoy the process of accomplishing our goals. So my advice to you in attacking the following exercises is go slow, check and re-check your steps every so often, learn to enjoy the process of trying a trail, experimenting, failing, trying another trail, and worry more about creating a valid trail than always getting the conclusion. This attitude should be especially useful for problems 9 (relatively difficult for this stage) and 10 (a little 7-step killer).



Step 6

Construct a formal proof of validity for each of the following arguments.

1.
 1. $A \supset \sim B$
 2. $C \supset B \therefore A \supset \sim C$
2.
 1. $D \supset (E \vee F)$
 2. $\sim E \bullet \sim F \therefore \sim D$
3.
 1. $(J \vee K) \supset \sim L$
 2. $L \therefore \sim J$
4.
 1. $X \equiv \sim P \therefore \sim P \equiv X$
5.
 1. $A \supset (B \bullet \sim C)$
 2. $H \vee (D \bullet A)$
 3. $\sim H$
 4. $T \supset C \therefore T \supset X$
6.
 1. $\sim B \vee [(C \supset D) \bullet (E \supset D)]$
 2. $B \bullet (C \vee E) \therefore D$
7.
 1. $\sim A \equiv (B \bullet C)$
 2. $(A \vee B) \supset T \therefore T$
- 8.*
 1. $(X \bullet Y) \supset Z$
 2. $\sim Z$
 3. $(T \vee P) \supset X$
 4. $Y \therefore T \supset P$
9.
 1. $X \supset (T \bullet P)$
 2. $T \supset X$
 3. $\sim(X \bullet T) \therefore \sim X$
- #10
 1. $B \therefore A \vee \sim A$

Brief Truth Tables Revisited and Decision Strategies

Hopefully in some of the above proofs you recognized subroutines that we have covered and were able to cut down on producing rambling trails. The transposition-hypothetical syllogism routine can be used in problem 1, some double negation bookkeeping can be used in problem 3, the commutation-simplification routine can be used in 5 and 6, the equivalence routine in 7, and the addition-De Morgan or the implication-distribution routine used in 9. As noted above, the recognition of subroutines allows us to experiment faster.

Yet in more difficult problems a situation is often produced where you have created so many trails that you begin to suffer sensory overload. Because you can take any piece of a reasoning trail and try to combine it with a piece of another trail, the possible combinations and choices become overwhelming. Sometimes, however, you may have a hunch that combining a part of one trail with a part of another trail will be fruitful, but because you can't be sure it will lead anywhere, you may be hesitant to try it because you might be wasting your time. It would be nice if a decision procedure existed that would let you know ahead of actually producing a proof that a trail will lead to the conclusion using the parts you have in mind.

There is such a procedure. Although this decision procedure will not construct a proof for you, it will give you the assurance that a proof exists. Actually, we have already studied this procedure in Chapter 8 in the section ***Brief Truth Tables***. Consider how a brief truth table could be used in the following situation.

- 8.
- 9.
- 10. $A \supset \sim X$
- 11.
- 12.
- 13.
- 14.
- 15. $B \supset \sim X$
- 16.

Suppose the conclusion you are attempting to derive is $B \supset A$ and in looking at lines 10 and 15 you have a hunch that it can be derived from these lines. Essentially, we are asking the question whether the following argument is valid.

- 1. $A \supset \sim X$
- 2. $B \supset \sim X \quad \therefore B \supset A$

If the answer is yes, then we know that a proof exists for deriving the conclusion from these premises alone, and we need not concern ourselves with any other lines or combination of lines. If the answer is no, then we know that it would be a waste of time to attempt a derivation of $B \supset$

A from these steps alone. Following the steps listed in the Brief Truth Table section of Chapter 8, we see that this argument is invalid.

T	1. $A \supset \sim X$		A=F
	F T		B=T
			X=F
T	2. $B \supset \sim X$	$\therefore B \supset A$	Invalid
	T T	T F	
		F	

By making the conclusion false, we can consistently make both premises true. So it would be a waste of time to attempt to derive the conclusion from steps 10 and 15 alone. These steps might be involved, but we now know for sure that at least some other steps must also be involved.

On the other hand, consider this situation where our hunch is that a conclusion $\sim A \vee \sim B$ can be derived using only lines 10, 13, and 15.

- 8.
- 9.
10. $A \supset \sim X$
- 11.
- 12.
13. $Y \supset X$
- 14.
15. $\sim Y \supset \sim B$
- 16.

Here is what a brief truth table would look like.

T	1. $A \supset \sim X$		A=T
	T T		B=T
			X=F
F	2. $Y \supset X$		Y=T
	T F		Valid
T	3. $\sim Y \supset \sim B$	$\therefore \sim A \vee \sim B$	
	F F	F F	
		F	

Because it is not possible to make the conclusion false and all the premises true,¹³ we know that

¹³ This time when we make the conclusion false, we cannot consistently make all the premises true. If we make X

$\sim A \vee \sim B$ can be derived validly from only these premises. Hence, in the above proof we would then concentrate on only lines 10, 13, and 15 and ignore all the other lines and possible combinations with those lines.

Because brief truth tables can be done quickly on scratch paper, you can experiment with various combinations of lines in difficult proofs prior to attempting any derivations. You will avoid a great deal of frustration if you know ahead of trying that a particular trail will not work. In problem 10 in the above Step 6 exercises, students will often spend considerable time attempting to derive a $\sim B$ based on the following hypothesis as to how this problem is done.

1. $B \quad \therefore A \vee \sim A$
2. $B \vee A \quad (1) \text{ Add}$
3. $\sim B \quad \text{????}$
4. $A \quad (2)(3) \text{ DS}$
5. $A \vee \sim A \quad (4) \text{ Add}$

Although it is admirable that they have obtained a level of proficiency to create such a hypothesis, a little reflection on the nature of a valid argument, put into the form of the simplest brief truth table, reveals that it is impossible to derive a $\sim B$ and that they are wasting their time even trying. Because we have only the premise B , deriving a $\sim B$ would be claiming that

1. $B \quad \therefore \sim B$

is a valid argument. But if $\sim B$ is false, then B is true, and we have an argument with a true premise and a false conclusion, the simplest and most obvious case of an invalid argument. Unless there is already a contradiction hidden in the premises, it will not be possible to derive one. So, any $\sim B$ in a line of proof for number 10 will indicate that there must be an incorrect step (X) somewhere in the proof.



Translations and Formal Proofs

Translate the following valid arguments, and when checked for accuracy by your instructor, then do formal proofs for each one.

1. If we buy the new furniture, then we can't purchase the new refrigerator provided that we buy the new carpet. If we buy the new furniture, we will buy the new carpet. We are

in premise 2 true to make this premise true, then premise 1 will become false.

going to buy the new furniture. So, we can't purchase the new refrigerator.

F = We buy the new furniture.
R = We purchase the new refrigerator.
C = We buy the new carpet.

- 2.* The new administration will appoint Shalayla to a cabinet post for the following reasons. Jordan and Adams will be appointed. Furthermore, Jordan being appointed is sufficient for Shalayla to be appointed, provided that Adams is also appointed.

S = The new administration will appoint Shalayla to the cabinet post.
J = Jordan will be appointed.
A = Adams will be appointed.

3. Argument for why prohibition measures don't work in stopping drug traffic.

Reduction by police of the supply of illegal drugs is a sufficient condition for increasing the price of these drugs. Potential income for drug traffickers will increase if prices rise, and if potential income for drug traffickers increases, it will produce an encouraging environment for drug trafficking. Thus, if the police reduce the supply of illegal drugs, it will produce an encouraging environment for drug trafficking.

S = Police reduce the supply of illegal drugs.
P = The price of illegal drugs increases.
I = Potential income for drug traffickers will increase.
E = An encouraging environment for drug traffickers will be produced.

4. If the president implements his tax programs, then the deficit will not continue to increase only if the economy turns around soon. The president will implement his tax programs. However, the economy will not turn around soon. Therefore, the deficit will continue to increase.

T = The president implements his tax programs.
D = The deficit will continue to increase.
E = The economy turns around.

5. Treat the first premise of the following argument as an **exclusive** disjunction.

We will hire either Aweau or Kaneshiro for the new electronics position. We will hire Aweau. So, we will not hire Kaneshiro. (A, K)

A = We will hire Aweau.
K = We will hire Kaneshiro.

6. The burglars entered the house from either the front or the rear. Had they entered from the rear the dog would have barked. But the dog did not bark. If the burglars entered the house from the front, then the maid must have let them in (because the alarm did not go off). This proves that the maid must have let them in.

F = The burglars entered the house from the front.
R = The burglars entered the house from the rear.
D = The dog barked.
M = The maid must have let the burglars into the house.

- 7.* The following statement is claimed to be true: "This statement is false." If the statement claims it is true and is true, then it is not true. However, if the statement claims that it is true but it is not true, then it is true. Hence, the statement is true if and only if it is not true. (C, T) Translate the first statement as C.

C = The statement, "This statement is false" is claimed to be true.
T = The statement, "This statement is false" is true.

8. Why backup lights on older VWs were wired through the transmission.

Backup lights are needed only if the car is in reverse. Because they are connected to the transmission they are on if and only if the car is in reverse. If they are on only if the car is in reverse, then they use very little power. Thus, the backup lights are on if needed, yet they use very little power.

N = Backup lights are needed
R = The car is in reverse.
O = Backup lights are on.
L = Backup lights use very little power.

9. Counselor to a student named John, "The probation policy is clear, if you don't achieve at least a 2.0 GPA or you don't complete at least 50% of all credits attempted this semester, you will be on probation next semester." Suppose the following semester John is not on probation. Using the probation policy as a premise and the premise that John is not on probation, construct a proof for the conclusion that John achieved at least a 2.0 GPA and completed 50% of all his credits attempted.

A = John achieved at least a 2.0 GPA.
C = John completed at least 50% of all credits attempted.
P = John is on probation.

10. Lee did not pass the final, but she did pass the course. So, it is not true that her passing the final was a necessary condition for her passing the course. (F, C)

F = Lee passed the final.
C = Lee passed the course.

- 11.* Famous argument attempting to prove that we are more than just a body of physical matter; that each of us has a non-physical mind as well.

If brain processes and mental occurrences are identical—that is, if all our thoughts are just physiological processes in the brain—then the former having spatial location constitutes a necessary and sufficient condition for the latter having spatial location. No one disputes that brain processes are spatially located. However, if mental processes are spatially locatable, then it will be meaningful to assign spatial location to thought. It is not meaningful to do this. So, it is not true that brain processes and mental occurrences are identical.

I = Brain processes and mental occurrences are identical.
B = Brain processes are spatially locatable (or have spatial location).
M = Mental processes are spatially locatable (or have spatial location).
T = It is meaningful to assign spatial location to thought.

12. An important issue in theology is known as the problem of evil. If God exists and is good, then why do humans suffer so much? Why so much cruelty to good people? To innocent children? Here is a valid argument that expresses this challenge. Construct a formal proof.

If God is good, then he must wish to abolish all evil; and if he is all-powerful, he must be able to abolish evil. If God wishes to abolish evil and is able to do so, then evil should not exist. But evil does exist. Therefore, God cannot be both omnipotent and good.

G = God is good.
W = God must wish to abolish all evil.
P = God is all-powerful (omnipotent).
A = God is able to abolish all evil.
E = Evil does exist.

If a fourth premise is added to this argument that "God is good," prove the conclusion, "God is not able to abolish all evil and He is not omnipotent."

13. Unless people outside Korea put pressure on the Japanese government, they will not listen to the women's ("comfort" women survivors) demands. If people outside Korea put pressure on the Japanese government, then the history books used in Japanese schools will be rewritten. Thus, either people outside Korea put pressure on the Japanese government and the history books used in Japanese schools are rewritten or the Japanese government will not listen to the women's demands.

- P = People outside Korea put pressure on the Japanese government.
- L = The Japanese government will listen to the comfort women's demands.
- H = The history books used in Japanese schools will be rewritten.

14. Below is an example of Kevorkian logic of suicide for a patient with inoperable, terminal cancer. (Please note that although this is a valid argument, it contains very controversial premises and assumptions. Do you have a right to take your own life? Many religions say you do not. Do not miracle cancer remissions sometimes occur? Is not the attitude or outlook on life of a patient important for the possibility of recovery?)

If you continue to take the cancer treatment, you will most likely still die a slow painful death or at least be a slave to pain the rest of your life. If you lie to yourself, the truth of your situation will not change; and, if the truth of your situation does not change, then you will most likely still die a slow painful death or at least be a slave to pain the rest of your life. You can either continue to take the cancer treatment or lie to yourself, or commit suicide. Thus, committing suicide is the only way for you to avoid a slow painful death or being a slave to pain the rest of your life.¹⁴

- C = You continue to take the cancer treatment.
- D = You will most likely still die a slow painful death.
- P = You will be at least a slave to pain the rest of your life.
- L = You lie to yourself.
- T = The truth of your situation changes.
- S = You commit suicide.

15. Happy-birthday card logic: "WHY WORRY???"

There are only two things to worry about, either you're healthy or you're sick. If you're healthy, there's nothing to worry about, and if you're sick, there are two things to worry about, either you'll get well or you won't. If you get well there is nothing to worry about, but if you don't, you'll have two things to worry about, either you'll go to heaven or to hell. If you go to heaven you have nothing to worry about, and if you go to hell you'll be so busy shaking hands with all of us that you'll have no time to worry. So, either you have nothing to worry about or you'll have no time to worry.

- H = You are healthy
- S = You are sick.
- W = You have something to worry about.
- G = You will get well.
- A = You will go to heaven.
- B = You will go to hell.

¹⁴ In this context, the best translation for "only way" is as an expression indicating a necessary condition.

T = You will have time to worry.

16. On being smart about global warming?

If the science on human influenced global warming is right, then we should act as if the scientific evidence on human influenced global warming is right, and if we act as if the scientific evidence on human influenced global warming is right, then we will have greater energy security. This is so, because acting as if the scientific evidence on human influenced global warming is not right will cause us to produce concentrations of carbon dioxide carrying immense risk, if the science on human influenced global warming is right. Acting as if the scientific evidence on human influenced global warming is right will cause us to invest in developing low-carbon emission technologies, even if the scientific evidence on human influenced global warming is not right. Investment in developing low-carbon emission technologies will cause greater energy security and energy efficiencies. We don't want to produce concentrations of carbon dioxide carrying immense risk, even though we act as if the scientific evidence on human influenced global warming is not right.¹⁵

H = The science on human influenced global warming is right.

R = We act as if the scientific evidence on human influenced global warming is right.

S = We have greater energy security.

C = We produce concentrations of carbon dioxide carrying immense risk.

L = We invest in developing low-carbon emission technologies.

E = We have greater energy efficiencies.

17. A government Catch 22?

The Islamic charity cannot sue the government unless members of the charity knew they were being wiretapped, but they could not know they were being wiretapped if and only if the wiretapping was legally secret. However, the government seems to be saying that even if the wiretapping was not legally secret, the charity could not know. If members of the charity knew (because they found out), the Islamic charity can sue the government only if the wiretapping was not legally secret. Hence, if the members of the charity knew or not that they were being wiretapped, the charity cannot sue the government.

C = The Islamic charity can sue the government.

K = The members of the charity knew they were being wiretapped.

S = The wiretapping was legally secret.

18. A philosophical position on whether humans are free and morally responsible for our actions.

¹⁵ Hint: X will cause Y should be translated as $X \supset Y$.

If the actions of human beings are completely determined by genetic as well as environmental factors, then humans cannot be morally responsible for their actions; whereas, if the actions of human beings are completely free, then there is no moral determinate for actions. It follows that, if either the actions of human beings are completely determined by genetic as well as environmental factors or the actions of human beings are completely free, then humans cannot be morally responsible for their actions or there is no moral determinate for actions.

D = Actions of human beings are completely determined by genetic as well as environmental factors.

R = Humans are morally responsible for their actions.

F = The actions of human beings are completely free.

M = There is a moral determinate for actions.

19. Conservative historian Victor David Hanson, "Weapons of Mass Hysteria," *National Review*, February 6, 2004. Good summary of the many issues.

"If the United States went to war with Iraq only because of the threat of WMDs; if the mass murdering of Saddam Hussein was found on examination to be highly exaggerated; if we had some secret plan for stealing the oil of Iraq, if Saddam Hussein posed no future threat to the United States or its allies; if the war resulted in a worse future for Iraq, the United States, and the surrounding Middle East; and if the administration deliberately constructed false intelligence evidence to advance such an unnecessary war that resulted in misery rather than hope, then an apology is needed now. But so far, that has simply not been the case."

Assuming "that has simply not been the case" means "no apology is needed now" to make a valid argument, translate the above as premises for the **conclusion**:

Either it is not true that the administration deliberately constructed false intelligence evidence to advance an unnecessary war that resulted in misery rather than hope, or it is not true that the United States went to war with Iraq only because of the threat of WMDs, or not true that the mass murdering of Saddam Hussein was found on examination to be highly exaggerated, or not true that we had some secret plan for stealing the oil of Iraq, or Saddam Hussein did pose a future treat to the United States or its allies, or the war did not result in a worse future for Iraq or the United States, or the surrounding Middle East.

W = The United States went to war with Iraq only because of the threat of WMDs.

E = The mass murdering of Saddam Hussein was found on examination to be highly exaggerated.

S = We had some secret plan for stealing the oil of Iraq.

T = Saddam Hussein posed a future threat to the United States.

- A = Saddam Hussein posed a future threat to the allies of the United States.
- I = The war resulted in a worse future for Iraq.
- U = The war resulted in a worse future for the United States.
- M = The war resulted in a worse future for the surrounding Middle East.
- C = The administration deliberately constructed false intelligence evidence to advance an unnecessary war that resulted in misery rather than hope.
- A = An apology is needed now.

20. Dilemma for President Obama in 2011.

President Obama will not be a one-term president if and only if he is not the first Democratic president that compromises with the Republicans on every single issue. If he is neither a one-term president nor the first Democratic president to compromise with the Republicans on every single issue, then he will not fulfill his campaign pledges. So, if he is either not a one-term president or the not the first Democratic president that compromises with the Republicans on every single issue, then he will not fulfill his campaign pledges.

- O = President Obama will be a one-term president.
- F = President Obama is the first Democratic president that compromises with the Republicans on every single issue.
- C = President Obama will fulfill his campaign pledges.

Holiday Adventures

The following two problems will give you a lot to worry about. They are quite challenging, requiring rather long valid trails for an introductory course in logic. In working on them, you may never obtain the desired conclusion. However, you will learn a lot about discipline (avoiding invalid steps to obtain the conclusion), and you will be immersing yourself into the agony and ecstasy, trial-and-error-trail-making, hypothesis-refutation-new-hypothesis process of symbolic reasoning that is so much a part of our scientific-technological culture. If nothing else, you will become very intimate with the nineteen rules and be able to do well on any exam your instructor gives you.

Challenging

A translation from Ex. III in Chapter 7. Approximately 30 steps!

1. $\sim G \supset \sim(A \vee B)$
2. $G \supset (H \supset C)$
3. $(H \bullet E) \supset \sim C$

$$4. B \supset (H \bullet E) / \therefore \sim B$$

Super Challenging!!!

This proof shows that association works for *if and only if*. Approximately 50 steps!!

$$1. C \equiv (D \equiv E) / \therefore (C \equiv D) \equiv E$$

Clarification Exercises

Using the rules of replacement transform and then rewrite the following sentences into simpler English sentences.

Example:

If it is not true that marijuana is not widely used or as harmful as alcohol, then it should be decriminalized. (W, H, D) $\sim(\sim W \vee H) \supset D$

$$\sim(\sim W \vee H) \supset D \gg (\sim\sim W \bullet \sim H) \supset D \text{ (DeM.) } \gg (W \bullet \sim H) \supset D \text{ (DN)}$$

If marijuana is widely used and not as harmful as alcohol, then it should be decriminalized. $(W \bullet \sim H) \supset D$

1. North believed it was not illegal not to tell Congress the truth. (L = Legal to lie to Congress.) $\sim\sim L$
2. It is not true that if North did not tell the truth to Congress, then he did tell the truth to Congress (T). $\sim(\sim T \supset T)$ Did North tell the truth to Congress or not?
3. Rain and humidity are not uncommon in Hawaii (R, H). $\sim\sim R \bullet \sim\sim H$
4. It's noisy here even when it is quiet (Q). $Q \supset \sim Q$
5. If it was ever the common law that by marriage (a women) gave irrevocable consent to sexual intercourse with her husband, it is no longer the common law (C). $C \supset \sim C$
6. It is not true that you can pass the final exam and not pass the course (F, C). $\sim(F \bullet \sim C)$ Transform into and rewrite as an "if...then" statement.
7. The view of the arbitration board that the fine against Johnson was justified but the suspension from the team was not is mistaken (F, S). $\sim(F \bullet \sim S)$ Transform into and

rewrite as an "if...then" statement.

8. It is not true that the accused accomplice not being in the neighborhood on the night of the crime is a sufficient condition for not having knowledge of the crime (N, K). $\sim(\sim N \supset \sim K)$ Transform into a conjunction (use *but* in the rewrite) and begin the rewrite, "It is possible that the accused accomplice . . . "
9. It is not true that having true premises is a sufficient condition for a valid argument (T, V). $\sim(T \supset V)$ Transform into a conjunction and begin the rewrite, "It is possible to have . . . "
10. It is not true that if the president implements his tax programs, then deficits will be reduced provided that the economy turns around (T, D, E). $\sim[T \supset (E \supset D)]$ Transform into a series of conjunctions and use *but* and *even though* in the rewrite.

Answers to Starred Exercises:

Step 4

4. **X** We can't add single negations to the parts of a conditional unless the antecedent and the consequent are reversed as in transposition.
9. **X** Commutation can be applied to part of a line, but that part can't be detached in the process. The correct application of commutation in #9 would be:

$$\begin{aligned} & (A \vee B) \supset (X \supset Y) \\ & \therefore (B \vee A) \supset (X \supset Y) \end{aligned}$$

18. Implication

21. **X** As noted above, we cannot apply DN to the premise.

Step 5

- 3.
- | | |
|---|-----------|
| 1. $(A \vee T) \supset [J \bullet (K \bullet L)]$ | |
| 2. $T \therefore J \bullet K$ | |
| 3. $T \vee A$ | (2) Add. |
| 4. $A \vee T$ | (3) Com. |
| 5. $J \bullet (K \bullet L)$ | (1)(4) MP |
| 6. $(J \bullet K) \bullet L$ | (5) Ass. |
| 7. $J \bullet K$ | (6) Simp. |

Step 6

- 8.
- | | |
|-------------------------------|-----------|
| 1. $(X \bullet Y) \supset Z$ | |
| 2. $\sim Z$ | |
| 3. $(T \vee P) \supset X$ | |
| 4. $Y \therefore T \supset P$ | |
| 5. $\sim(X \bullet Y)$ | (1)(2) MT |
| 6. $\sim X \vee \sim Y$ | (5) De M. |
| 7. $\sim Y \vee \sim X$ | (6) Com. |
| 8. $\sim \sim Y$ | (4) DN |
| 9. $\sim X$ | (7)(8) DS |

- | | |
|-----------------------------|------------|
| 10. $\sim(T \vee P)$ | (3)(9) MT |
| 11. $\sim T \bullet \sim P$ | (10) De M. |
| 12. $\sim T$ | (11) Simp. |
| 13. $\sim T \vee P$ | (12) Add. |
| 14. $\sim\sim T \supset P$ | (13) Impl. |
| 15. $T \supset P$ | (14) DN |

Translations:

2.

1. $J \bullet A$
2. $A \supset (J \supset S) \therefore S$

7.

1. C
2. $(C \bullet T) \supset \sim T$
3. $(C \bullet \sim T) \supset T \therefore T \equiv \sim T$

11.

1. $I \supset (B \equiv M)$
2. B
3. $M \supset T$
4. $\sim T \therefore \sim I$

Essential Logic
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