

## CHAPTER 9: SYMBOLIC TRAILS AND FORMAL PROOFS OF VALIDITY, PART 1

### *Introduction*

Throughout this book we have used the metaphor of a "reasoning trail." The cultural roots for our use of logic and mathematics can be traced back to the ancient Greeks. The ancient Greeks believed that our reasoning ability gave us a special mystical power to "see" or detect unseen realities. They thought we could start with what we immediately experience and then follow a trail using our logic and/or mathematics to transport our minds places inaccessible to our immediate experience. Thus, the blind man was able to see in a sense that he had on a white hat, just as Eratosthenes was able to "see" the size and shape of the Earth even though he was visually limited to a small piece of our large Earth. What gave the Greeks and much of our past Western culture confidence that we indeed had this power was the metaphysical belief that there was a resonance between our thinking and reality. Reality was thought to have particular trails or laws, and when we think correctly it was thought that we are mirroring those trails or laws.<sup>1</sup>

Most modern philosophers no longer accept this metaphysics, but see our logic and mathematics instead as human constructions that we impose on reality, as practical tools that we use to successfully interface or work with reality. Although we may no longer possess the same confidence in our intellectual specialness -- the confidence that somehow God has given us a head start by supplying us with the same thoughts with which He has constructed reality -- the results of this initial confidence are with us today as never before. For good or ill, we live in a scientific-technological culture, where people daily sit down in front of desks and follow or analyze reasoning trails symbolically. Most often this is now in the form of computer programs or with the application of computer programs, but whether using computers or not the process is the same: Using assumptions based on accepted knowledge, information technology specialists attempt to design the most efficient Internet connections and networks; the engineer wants to see or discover prior to actual construction how a bridge will look and function; using the laws of nature, the physicist wants to see the course of a space craft and the navigational

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<sup>1</sup> For a summary and critique of this past cultural perspective, see Richard Rorty's, *Philosophy and the Mirror of Nature* (Princeton, New Jersey: Princeton University Press, 1979).

corrections that will be necessary to keep it on course when it encounters gravitational influences; the chemist wants to see the properties and use of a new combination of known elements, and so on.

In this chapter and the next you will be matching these processes of following a symbolic reasoning trail. You will be learning how to create your own symbolic reasoning trails, and you will be experiencing the adventure of symbolic problem solving: the trial and error, the back-to-the-drawing-board frustration and play, the tension of a problem not solved, and hopefully, at least some of the time, the feeling of completion when the trail ends successfully. Most people come to enjoy this in spite of their initial fear of mathematics and any kind of symbolic reasoning. There is something in our nature that makes us enjoy problem-solving, that makes us enjoy being a detective, of being in the "hunt" so to speak of a solution to a puzzle, of seeking something just out of our grasp. At night you will even dream of solutions to some of the symbolic problems we will be working on. Don't worry about this when it happens; your brain loves this stuff.<sup>2</sup>

Truth tables are mechanical; the process of symbolic reasoning that we will be learning in this chapter is not. This process will involve creativity, discipline, and perseverance, and for this reason individual personality factors will emerge as one is tested by the adversity of not knowing in any mechanical fashion the right path to take. We will need to take some time and learn this new process in several steps. For an overview of where we will be going, consider the following example.

For most of us when we problem-solve or analyze we must slow our thinking down and examine possible trails piece by piece. Occasionally we meet individuals who have something like photographic minds and the special ability to keep everything focused, seeing where every step leads instantly. When Ronald Reagan was president of the United States I once went to a lecture by a government expert on the implications of what was called then the "Iran-Contra" scandal. Ronald Reagan had been elected in 1980 in part by convincing the voters that he would be a stronger leader than Jimmy Carter. During the last stages of the Carter presidency, U. S. embassy staff were being held hostage in Iran and all the U. S. military power and influence seemed impotent in being able to do anything to stop this injustice and embarrassment. Further, this weakness seemed to coincide alarmingly with an increasing economic weakness for the U. S. in the world order. Carter was portrayed by Republicans in 1980 as a "wimp"; Reagan was the "Duke," the hero in a John Wayne movie, the Marlboro man riding off into the sunset after dealing with all the bad guys. With Reagan the U. S. would not let terrorists push us around anymore.

So it was very embarrassing for Reagan, during his second term in office, to admit that he had negotiated with terrorists. The Reagan administration had promised never to

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<sup>2</sup> It will not be permanent! Within a few weeks after your logic course is over the dreams will cease.

negotiate with terrorists because this would legitimize and encourage their illegal and immoral actions. However, at that time Reagan admitted to the American people that he had allowed the sale of military equipment to Iran in exchange for their influence in getting U. S. hostages in Lebanon released.<sup>3</sup> More serious than just politically embarrassing was the revelation that the proceeds from the sale of arms to Iran were used to support the Nicaraguan Contras. The sale of arms to Iran may have been embarrassing or stupid, but supporting the Contras militarily was strictly illegal -- Congress had passed a law prohibiting the U. S. government from supporting the Contras other than with humanitarian aid. Reagan claimed never to have known about the Contra diversion; if he did he would have been impeached, because as a government of law, not of men, even the president cannot violate the law without sanction. In fact, it is one of the president's principal duties to make sure all the laws of the land are upheld.

For about an hour I listened to the government expert analyze the various political and legal ramifications of the Iran-Contra scandal. He talked about constitutional issues and precedents, the various administration officials involved and their responsibilities, the U. S. foreign policy, the evidence for and against whether Reagan knew about the Contra diversion of funds, and the implications of impeachment. At the end of the talk there was a question and answer discussion session with the audience. At one point a man stood up and announced very confidently that the most immediate implication of all that the speaker had said was that Ed Meese, the Attorney General for Reagan at this time, should resign from office. The speaker seemed a little stunned. He had spoken for over an hour, painstakingly analyzing detail after detail, conveying years of experience and reflection on government matters, and the questioner had the audacity to state that everything boiled down to one simple implication. But the speaker was intrigued; there was something in the implication that seemed appealing. Like a nibble at the end of a fishing line, something that needed to be pulled out of the muddy water. So he asked the questioner to elaborate, and the man responded with a quick summary that went something like this.

Well it is clear that Reagan lied about the Iran deal. He has admitted this to the American people and asked for their forgiveness, explaining how important it was for him to win the release of U. S. hostages. But if he lied about the Iran deal, then, as you have explained, he also lied about the Contra deal or he should have known about the Contra deal. The buck stops with the president and he is responsible for guaranteeing laws are not broken, especially by his own staff. Now Reagan claims not to have known about the Contra deal. Let's assume this is true. Well, since the chief of staff is responsible for the flow of information to the president (He is actually one of the most powerful persons in government, and

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<sup>3</sup> During this time Iran and Iraq were engaged in a bloody and futile war of human attrition, and we were also supporting Iraq militarily.

not elected.), he should assume responsibility for the president not knowing about the Contra diversion and resign. McFarlane, who has obviously been the scapegoat, should be exonerated. And finally, since the Attorney General (Ed Meese) is responsible for ensuring that all laws are followed, the necessary conditions of responsibility that apply to the chief of staff also apply to Meese. So, Meese should resign immediately.

As the man's rapid fire logic cascaded about the room, eyebrows were raised, and the speaker seemed to be getting embarrassed. When he finished there was a hushed silence.

What would the speaker say? The man's reasoning seemed to flow; especially since he had put it together so quickly, meshing the thoughts together one by one like bricks in a sturdy cemented wall. His conclusion seemed like a novel idea that was hiding behind the complexity of a million facts, now pulled out for all to see. But was his insight correct? The speaker gave an answer, the specifics of which I don't remember. It was obvious that he did not know what to say, and he basically gave the type of answer that changed the subject and evaded the issue.

In the immediate flow of experience it is often hard to hold on to all the details, to isolate what is important and arrange what is important as premises for a reasoning trail. So the questions and discussion jumped around, moving on to other topics raised by the speaker, and the subject of Meese resigning did not come up again that night. But I suspect that when the speaker had more time to think about what the questioner had said, he began to talk with his friends about the necessity of Meese resigning. Let's slow our thinking down also and analyze the man's logic step by step. First, a more formal presentation of his argument.

1. If Reagan lied about the Iran deal (I), then he either lied about the Contra deal (C) or he should have known about the Contra deal (K).
2. It is clear he did lie about the Iran deal.
3. But (let's assume) he did not lie about the Contra deal.
4. If he should have known about the Contra deal, then his chief of staff (S) should not continue in office and McFarlane should be exonerated (E).
5. Also, Meese should continue in office (M) only if Reagan's chief of staff continues in office.

Therefore, Meese should not continue in office.

Next a translation of the argument.

1.  $I \supset (C \vee K)$
2. I

3.  $\sim C$
4.  $K \supset (\sim S \bullet E)$
5.  $M \supset S \quad \therefore \sim M$

Now if we combine some of these premises in isolation, we can derive some "mini" conclusions by just using our common sense. The first premise says that if I is true, then  $C \vee K$  is true. Since the second premise says that I is true, we can conclude that  $C \vee K$  is true. So,

### Step 1

$$\begin{array}{l} I \supset (C \vee K) \\ I \quad \therefore C \vee K \end{array}$$

But the third premise says that C is not true. Since we have concluded that  $C \vee K$  is true, but now know that C cannot be true, then we can conclude that K is true. So,

### Step 2

$$\begin{array}{l} C \vee K \\ \sim C \quad \therefore K \end{array}$$

But the fourth premise says that if K is true, then S is not true and E is true. Since we now know that K follows from the previous premises, we can conclude that S is not true and E is true as follows,

### Step 3

$$\begin{array}{l} K \supset (\sim S \bullet E) \\ K \quad \therefore \sim S \bullet E \end{array}$$

Well, if we know that S is *not* true and E is true, then we know that S is not true. If we know two things, then we surely know one thing. So,

### Step 4

$$\begin{array}{l} \sim S \bullet E \\ \therefore \sim S \end{array}$$

Finally, the last premise says that M is true only if S is true. So, since we have discovered that S is not true, M is not true.

## Step 5

$M \supset S$   
 $\sim S \quad \therefore \sim M$

Reflect now on what we have done. We have shown that if the above premises (1-5) are true, then  $\sim M$  is true. In other words, we have shown this argument to be valid by creating a chain of reasoning in which each mini-step is valid, such that starting with the premises we created a number of steps until we arrived at the conclusion  $\sim M$ . Proving this argument to be valid using the truth table method would have required a complicated table with 64 lines! We have better things to do with our time.

## Constructing Formal Proofs of Validity

What you will be learning to do in this chapter is to formalize the method of proof of validity that we just used. We will be learning a rigorous way of presenting our common sense. We will be creating reasoning trails, such that complicated arguments will be proved to be valid by creating a chain of *elementary valid arguments*. Here is an example of what this rigorous method of presentation will look like applied to the above argument.

- |                                   |                     |
|-----------------------------------|---------------------|
| 1. $I \supset (C \vee K)$         |                     |
| 2. $I$                            |                     |
| 3. $\sim C$                       |                     |
| 4. $K \supset (\sim S \bullet E)$ |                     |
| 5. $M \supset S$                  | $\therefore \sim M$ |
| 6. $C \vee K$                     | (1)(2) MP           |
| 7. $K$                            | (6)(3) DS           |
| 8. $\sim S \bullet E$             | (4)(7) MP           |
| 9. $\sim S$                       | (8) Simp.           |
| 10. $\sim M$                      | (5)(9) MT           |

Notice that lines 6-10 list the steps we took above. This rigorous method of presentation is called a **Formal Proof**. Formal Proofs are simply objective methods of presentation of reasoning trails, such that anyone who learns the method of presentation can check the steps against their own common sense. Lines 6 through 10 show our chain of reasoning. The numbers adjacent to each line show the premises used to infer each line as a conclusion, as a link in the chain of reasoning. To derive line 6 as a valid conclusion we used premises 1 and 2 (Step 1 above). We then used line 6 with line 3 to derive line 7 (Step 2 above). Then putting line 7 together with line 4 we derived line 8 (Step 3). From 8 we knew 9 (Step 4), and finally from line 9 with line 5 we concluded line 10 (Step 5).

The capital letters next to the lines of justification refer to names of elementary valid arguments. We saw in Chapters 4 and 5 that fallacies have been named. Similarly, logicians have studied our common sense carefully and named many of our elementary common sense inferences. Notice that lines 6 and 8 have the same capital letters (MP) adjacent to the lines of justification. Although these lines involve different content (different letters), if you look carefully the **form** or pattern of reasoning is the same.

$$\begin{array}{ll} I \supset (C \vee K) & K \supset (\sim S \bullet E) \\ I \quad \therefore C \vee K & K \quad \therefore \sim S \bullet E \end{array}$$

$$\begin{array}{l} p \supset q \\ p \quad \therefore q \end{array}$$

**Modus ponens** is the fancy name that logicians give to this form of reasoning. This Latin name, which means to be in the mode of affirmation (of the antecedent), shows that this form of reasoning has been recognized to be valid for centuries. We know intuitively that it is an elementary valid argument and we saw in the last Chapter (**Argument Forms and Variables**) that a truth table also shows this pattern to be valid. Remember the time-saving virtue of form recognition. Complicated arguments such as **A3** in Chapter 8 can be seen to be valid at a glance. Recall that all of these arguments have the same form as MP.

#### A1 from C8

1.  $(M \bullet C) \supset \sim B$
2.  $M \bullet C \quad \therefore \sim B$

#### A2 from C8

1.  $P \supset C$
2.  $P \quad \therefore C$

#### A3 from C8

1.  $\{[(A \supset B) \equiv (C \vee \sim B)] \bullet \sim D\} \supset (\sim E \vee \sim F)$
2.  $\{[(A \supset B) \equiv (C \vee \sim B)] \bullet \sim D\} \quad \therefore \sim E \vee \sim F$

Once we know that the form of an argument is valid, we know an infinite number of arguments to be valid. We know that any argument that fits the form is valid, just as once a child learns what a chair is, he or she can apply this concept to a multitude of

different things that all have in common the fact that they are chairs. For instance, the following argument has the same form as those above.

If BP says that they are containing now 90,000 barrels of oil a day from the Gulf of Mexico leak (C), then their original estimate that the leak was only 5,000 barrels a day was a deliberate lie (L) or a terrible estimate (T). BP says that they are containing now 90,000 barrels of oil a day from the Gulf of Mexico leak. So, their original estimate that the leak was only 5,000 barrels a day was a deliberate lie or a terrible estimate.

$C \supset (L \vee T)$   
 $C \quad \therefore (L \vee T)$

It did not require rocket science math for most people to see that there was something wrong with all the PR stories BP was telling people about what was going on in the spring and summer of 2010.

### **Step 1: Recognizing Forms -- Copi's 9 Rules of Inference**

We are now ready for the first step in constructing formal proofs. Lines 7, 9, and 10 of the formal proof above have the justifications DS, Simp, and MT. These abbreviations stand for disjunctive syllogism, simplification, and modus tollens. Along with modus ponens, we must examine and learn the forms of these elementary common sense inferences as well as five other rules. We will call these rules **Copi's Nine Rules of Inference** after the logician Irving Copi, who was the first to systematize these rules in textbook form for previous generations of logic students for constructing formal proofs.<sup>4</sup> What follows is a presentation of each rule with three examples of application. Your task is to continue the form-recognition process that we began at the end of Chapter 8. Examine each of the three examples presented for each rule and make sure you see how each fits the argument form presented with variables. If you do not see how any example fits the argument form of a rule, mark it and ask your instructor to explain the fit.

### **MODUS PONENS (MP)**

For the sake of completeness and to be able to compare the valid form of modus ponens with the form of a very common fallacy, let's review this rule one more

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<sup>4</sup> See Copi's classic *Introduction to Logic*, 8th ed. (New York: Macmillan Publishing Co., 1990).

time.

| Modus Ponens          |                                   |                                                                                     |
|-----------------------|-----------------------------------|-------------------------------------------------------------------------------------|
| <b>Egs. #1</b>        | $A \supset B$<br>$A \therefore B$ | <b>#2</b> $R \supset J$<br>$R \therefore J$                                         |
|                       |                                   | <b>#3</b> $\sim(I \equiv R) \supset \sim P$<br>$\sim(I \equiv R) \therefore \sim P$ |
| <b>Argument Form:</b> | $p \supset q$<br>$p \therefore q$ |                                                                                     |

Although it is better to just see the pattern, to help you recognize an application of modus ponens in the future, reflect on the essence of this rule. All three examples above have two premises, one of the premises has ( $\supset$ ) as a major connective, the other premise matches exactly the antecedent of the first premise, and the conclusion matches the consequent of the first premise exactly. Note that it does not matter what the antecedent or consequent are, they can be simple, complex, or involve negations as in number 3. All that matters is that they match in this way.

Here are three more arguments. Do they match the pattern of modus ponens?

|                                      |                                                              |                                                                        |
|--------------------------------------|--------------------------------------------------------------|------------------------------------------------------------------------|
| #1 $A \supset B$<br>$B \therefore A$ | #2 $(S \bullet R) \supset T$<br>$T \therefore (S \bullet R)$ | #3 $\sim H \supset (P \bullet D)$<br>$(P \bullet D) \therefore \sim H$ |
|--------------------------------------|--------------------------------------------------------------|------------------------------------------------------------------------|

You should have answered "no" in all three cases. The second premise does not match the antecedent of the first premise; instead it matches or "affirms" the consequent. Furthermore, the conclusion matches the antecedent rather than the consequent as in modus ponens. The argument form that fits all three examples is

|                  |           |
|------------------|-----------|
| $p \supset q$    | (Invalid) |
| $q \therefore p$ |           |

A truth table of this argument form will show it to be invalid. This form represents a very common mistaken inference. It is called the **Fallacy of Affirming the Consequent (FAC)** and should never be used in a formal proof. Its persuasiveness is no doubt caused by its close resemblance to modus ponens. For instance, the argument

If John passes the final exam, he will pass the course.  
John passed the course.  
Therefore, John passed the final.

might sound good and seem to flow, but as we have seen, the first premise does not specify that passing the final is the only way John can pass the course. The first premise specifies what would be sufficient for John to pass the course, but not what is necessary. Thus the conclusion could be false even if the premises are true. Compare this argument with

John will pass the course only if he passes the final.  
 John passed the course.  
 Therefore, John passed the final.

This is an example of modus ponens. We are locked into the conclusion, because if the first premise is true that passing the final is a necessary condition for passing the course, then since John passed the course, he must have passed the final.

## MODUS TOLLENS (MT)

Step 5 in the formal proof on Iran-Contra scandal above was an example of modus tollens (Latin for being in the mode of denying the antecedent in the conclusion). Here are some more examples followed by the argument form.

| Modus Tollens        |                                                |                                                                                  |
|----------------------|------------------------------------------------|----------------------------------------------------------------------------------|
| <b>Egs.</b>          | #1 $A \supset P$<br>$\sim P \therefore \sim A$ | #2 $\sim G \supset \sim(A \vee B)$<br>$\sim\sim(A \vee B) \therefore \sim\sim G$ |
|                      |                                                | #3 $S \supset L$<br>$\sim L \therefore \sim S$                                   |
| <b>Argument Form</b> | $p \supset q$<br>$\sim q \therefore \sim p$    |                                                                                  |

Reflect on the essence of this rule. Modus tollens has two premises, one of which has ( $\supset$ ) as a major connective; the other premise negates whatever the consequent is of the ( $\supset$ ) premise, and the conclusion is always a negation of the antecedent of the ( $\supset$ ) premise. Note that although complex, example 2 stays true to the rule. The consequent of the ( $\supset$ ) premise is  $\sim(A \vee B)$ , so to be an example of modus tollens the second premise must be  $\sim\sim(A \vee B)$ . Also, the antecedent of the ( $\supset$ ) premise is  $\sim G$ , so the conclusion must be a negation of this or  $\sim\sim G$ .<sup>5</sup> See it?

<sup>5</sup> Although our common sense tells us that G is opposite of  $\sim G$ , technically G is not a negation of  $\sim G$ . The negation of a negation is a double negation ( $\sim\sim G$ ). So if G were the conclusion in example 2, the argument

$$\begin{aligned} p &= \sim G \\ q &= \sim(A \vee B) \end{aligned}$$

Examine the following arguments. Do they fit the form of modus tollens?

$$\begin{array}{lll} \#1 \ S \supset L & \#2 \ (A \vee B) \supset H & \#3 \ X \supset \sim(Y \supset A) \\ \sim S \therefore \sim L & \sim(A \vee B) \therefore \sim H & \sim X \therefore \sim(Y \supset A) \end{array}$$

You should have answered "no" in all three cases. Rather than negate the consequent of the first premise as in modus tollens, the second premise in all three cases negates or "denies" the antecedent of the first premise. Also, rather than concluding with the negation of the antecedent of the first premise, these arguments conclude with the negation of the consequent. The argument form for these arguments is

$$\begin{array}{ll} p \supset q & \text{(Invalid)} \\ \sim p \therefore \sim q \end{array}$$

and is called the **Fallacy of Denying the Antecedent (FDA)**. No doubt the persuasiveness of this form of reasoning is due to its closeness to modus tollens. But this argument form is invalid and should never be used in a formal proof. Consider these arguments.

If Dao-Ming lives in the city of Shanghai, she lives in China.  
 Dao-Ming does not live in the city of Shanghai.  
 Therefore, she does not live in China. (fallacy of denying the antecedent -- invalid)

and

If Dao-Ming lives in the city of Shanghai, she lives in China.  
 Dao-Ming does not live in China.  
 Therefore, Dao-Ming does not live in the city of Shanghai. (modus tollens -- valid)

The first example is invalid, because Dao-Ming might not live in Shanghai, but she could still live somewhere else in China. The second example (MT) is valid, because if she does not live anywhere in China, then she surely does not live in any city in China.

Again, although it is better to just focus on seeing these forms, let's summarize what we

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would be valid but not yet an example of modus tollens.

have learned about arguments involving conditionals. In the valid arguments MP and MT the premises involve a relationship whereby the antecedent is affirmed or the consequent is denied; whereas in the invalid arguments FAC and FDA the consequent is affirmed or the antecedent is denied.

## DISJUNCTIVE SYLLOGISM (DS)

Step 2 in the above formal proof was an example of disjunctive syllogism. Here are some more examples and the argument form.

| Disjunctive Syllogism |                                        |                                                                           |                                                               |
|-----------------------|----------------------------------------|---------------------------------------------------------------------------|---------------------------------------------------------------|
| <b>Egs.</b>           | #1 $C \vee K$<br>$\sim C \therefore K$ | #2 $\sim G \vee \sim(A \vee B)$<br>$\sim\sim G \therefore \sim(A \vee B)$ | #3 $(I \bullet P) \vee R$<br>$\sim(I \bullet P) \therefore R$ |
| <b>Argument Form</b>  | $p \vee q$ $\sim p \therefore q$       |                                                                           |                                                               |

Note that the essence of disjunctive syllogism is that one premise must be an (v) statement (a disjunction) and the other premise must negate the left hand side of the (v) statement. Then the conclusion exactly matches the right hand side of the disjunction. Note example 2. If the left hand side of the disjunction premise is  $\sim G$ , then to stay true to the rule the second premise must be  $\sim\sim G$ .

An argument that has a premise that negates the right hand side of the disjunction and concludes the left hand side of the disjunction would be valid, but would not yet be an example of disjunctive syllogism.<sup>6</sup> Disjunctive syllogism is valid for both inclusive and exclusive disjunctions. However, arguments of the form

$$p \text{ or } q$$

$$p \therefore \sim q$$

are invalid for inclusive disjunctions, but valid for exclusive disjunctions. Consider the following.

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<sup>6</sup> We say "not yet an example" because we will be combining our rules of inference with other rules to turn the example just given into a disjunctive syllogism.

This man must be either very intoxicated (drunk) or have diabetes.  
 He is very intoxicated.  
 Therefore, he must not have diabetes.  
 (Inclusive *or* so invalid; the man could be both intoxicated and have diabetes.)

1.  $I \vee D$
2.  $I \therefore \sim D$  (invalid)

We will hire either Aweau or Kaneshiro for the new electronics position.  
 We will hire Aweau.  
 So, we will not hire Kaneshiro.  
 (Exclusive *or* so valid; there is only one new position.)

If we did a truth table on the translation of this argument, we would find it to be valid.

1.  $(A \vee K) \bullet \sim(A \bullet K)$
2.  $A \therefore \sim K$  (valid)

The important point for now is that disjunctive arguments that have a premise that negates the right hand side of the disjunction, or ones that have a premise that matches the left hand side of a disjunction are not examples of disjunctive syllogism. To be an example of disjunctive syllogism, a premise must negate the left hand side of a disjunction, and the conclusion must be an exact match of the right hand side of the disjunction.

$p \vee q$   
 $\sim q \therefore p$  (valid, but not DS)

$p \vee q$   
 $p \therefore q$  (invalid and not DS)

## HYPOTHETICAL SYLLOGISM (HS)

Although not used in our formal proof, a common form of valid reasoning is to "chain" if-then ( $\supset$ ) statements together as in the following argument.

If the cold war is over, then less tax money can be spent on defense.  
 If less tax money can be spent on defense, then a tax reduction is possible and/or more money is available for rebuilding our infrastructure and supporting education.

So, since the cold war is over, then a tax reduction is possible and/or more money is available for rebuilding our infrastructure and supporting education.

The reasoning form for such if-then chaining is called hypothetical syllogism. Here are some more examples and the argument form. (The third example is a translation of the above argument.)

| Hypothetical Syllogism |                                                               |                                                                                                         |
|------------------------|---------------------------------------------------------------|---------------------------------------------------------------------------------------------------------|
| <b>Egs.</b>            | #1 $A \supset B$<br>$B \supset C$<br>$\therefore A \supset C$ | #2 $\sim(B \vee K) \supset K$<br>$K \supset H$<br>$\therefore \sim(B \vee K) \supset H$                 |
|                        |                                                               | #3 $C \supset L$<br>$L \supset [T \vee (I \bullet E)]$<br>$\therefore C \supset [T \vee (I \bullet E)]$ |
| <b>Argument Form:</b>  | $p \supset q$<br>$q \supset r \therefore p \supset r$         |                                                                                                         |

Note the essence of hypothetical syllogism. The major connective of both premises and the conclusion is ( $\supset$ ). The consequent of one premise matches the antecedent of the other premise. The conclusion then links the antecedent of one premise with the consequent of the other premise.

Two common invalid inferences that are often confused with hypothetical syllogism are:

|                                      |                  |     |                                      |                  |
|--------------------------------------|------------------|-----|--------------------------------------|------------------|
| $p \supset q$                        | <b>(invalid)</b> | and | $p \supset q$                        | <b>(invalid)</b> |
| $r \supset q \therefore p \supset r$ |                  |     | $q \supset r \therefore r \supset p$ |                  |

Neither should ever be used in a formal proof.<sup>7</sup>

## CONSTRUCTIVE DILEMMA (CD)

A more complicated form of valid reasoning than the rules presented thus far involves combining conditional statements with a conjunction and a disjunction. Here is an example.

<sup>7</sup> Number 14, Exercises III, of Chapter 1 is an example of the second invalid form. It is much easier to see that this argument is invalid on the basis of recognizing its form.

If we send more troops to Afghanistan, the population will hate us even more as occupiers of their country; whereas, if do not send more troops to Afghanistan, the Taliban will most likely take over the country. We either send more troops or we do not. So, either the population will hate us even more as occupiers of their country or the Taliban will most likely take over the country.

This form of valid reasoning is called constructive dilemma. Below are translated examples followed by the argument form. (The second example is a translation of the above argument.)

| Constructive Dilemma  |                                                                                 |                                                                                           |                                                                                                                                               |
|-----------------------|---------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------|
| <b>Egs.</b>           | #1 $(A \supset B) \bullet (C \supset D)$<br>$A \vee C$<br>$\therefore B \vee D$ | #2 $(S \supset H) \bullet (\sim S \supset T)$<br>$S \vee \sim S$<br>$\therefore H \vee T$ | #3 $[(A \bullet B) \supset \sim C] \bullet [\sim D \supset (X \vee Y)]$<br>$(A \bullet B) \vee \sim D$<br>$\therefore \sim C \vee (X \vee Y)$ |
| <b>Argument Form:</b> | $(p \supset q) \bullet (r \supset s)$<br>$p \vee r \therefore q \vee s$         |                                                                                           |                                                                                                                                               |

Although more complicated than any of our previous rules, with a little concentration you should be able to see the essence of constructive dilemma. In one sense, CD is like a double MP.

|                  |                  |
|------------------|------------------|
| $p \supset q$    | $r \supset s$    |
| $p \therefore q$ | $r \therefore s$ |

To be a constructive dilemma, however, we must have two premises, one of which is a conjunction ( $\bullet$ ) of two conditionals ( $\supset$ ), the other of which must be a disjunction ( $\vee$ ) of the two antecedents of the conditionals. Then the conclusion must be a disjunction ( $\vee$ ) of the two consequents of the conditionals. Number 3 is a good test of your form recognition ability. Notice that the major connective of the first premise is ( $\bullet$ ), and it connects two ( $\supset$ ) statements. The antecedents of these two conditionals are  $(A \bullet B)$  and  $\sim D$ , and these two statements are matched exactly and connected by a ( $\vee$ ) statement in the second premise. The consequents of the two conditional statements are  $\sim C$  and  $(X \vee Y)$ , and these are matched exactly and connected by a ( $\vee$ ) statement in the conclusion.

$p = A \bullet B$

$q = \sim C$   
 $r = \sim D$   
 $s = X \vee Y$

Remember that the **p**, **q**, **r**, and **s** are **variables** and can stand for anything whatsoever. It does not matter whether they stand for something simple or complex; all that matters is that the content of an argument matches up to fit the above form.

## CONJUNCTION (Conj.)

To be able to construct formal proofs, logicians have discovered that even our most trivial and obvious common sense inferences must be identified. It is a trivial, simple inference to say that if we know two things separately, then we know two things together.

If we know that Galileo was a religious man, and we also know that Kepler was a religious man, then we know that Galileo and Kepler were both religious men. In essence, if we know two separate premises to be true, then we know the conjunction of those premises is true. This rule of inference is called conjunction. Here are some translated examples followed by the argument form.

| Conjunction   |               |                |
|---------------|---------------|----------------|
| Egs. #1 A     | #2 K•G        | #3 G⊃L         |
| B             | T•B           | ~G⊃U           |
| ∴ A•B         | ∴ (K•G)•(T•B) | ∴ (G⊃L)•(~G⊃U) |
| Argument Form | p             |                |
|               | q ∴ p • q     |                |

Notice the simple essence of this rule. No matter what the premises are, the conclusion is a simple combination of these premises by the (•) connective. All this rule says is that you can combine premises by a conjunction.

We will now look at some rules that are also needed for the completeness of constructing formal proofs, but which have only one premise.

## ABSORPTION (Abs.)

Similar to the conjunction rule is the following obvious inference: If I know that passing the final is sufficient for John passing the course, then I know that if John passed the final, he passed both the final and the course. This common sense inference is called absorption. Here are some examples and the argument form. (The first one is a translation of the example just given.)

| Absorption            |                                                          |                                                                                          |
|-----------------------|----------------------------------------------------------|------------------------------------------------------------------------------------------|
| <b>Egs.</b>           | #1 $F \supset C$<br>$\therefore F \supset (F \bullet C)$ | #2 $J \supset \sim(S \bullet K)$<br>$\therefore J \supset [J \bullet \sim(S \bullet K)]$ |
|                       |                                                          | #3 $W \supset \sim W$<br>$\therefore W \supset (W \bullet \sim W)$                       |
| <b>Argument Form:</b> | $p \supset q$<br>$\therefore p \supset (p \bullet q)$    |                                                                                          |

Reflect on the essence of the absorption rule. It must have only one premise. Both the premise and the conclusion must have ( $\supset$ ) as a major connective, and the consequent of the conclusion must combine the antecedent and the consequent of the premise by the ( $\bullet$ ) connective.

## SIMPLIFICATION (Simp.)

Step 4 in the above formal proof involved the simple commonsense inference: If we know two things, then we know one thing. Since we had concluded that the chief of staff should resign and McFarlane should be exonerated, we can of course conclude that the chief of staff should resign. If we know that John passed the final and the course, then we know that John passed the final. This elementary inference is called simplification. Here are some examples and the argument form.

## Simplification

Egs.    #1  $\sim S \bullet M$     #2  $(L \supset D) \bullet (\sim D \supset L)$     #3  $(F \vee G) \bullet \sim E$   
               $\therefore \sim S$                  $\therefore L \supset D$                  $\therefore F \vee G$

Argument Form:                 $p \bullet q$   
                                               $\therefore p$

Please note carefully the essence of simplification. As elementary as this rule is, it is the most often abused rule by beginning logic students. This rule has only one premise, **the major connective of the premise must be a conjunction ( $\bullet$ )**, and the conclusion must be the first part of the conjunction. The following examples are common misapplications of simplification by beginning logic students.

#1  $\sim(F \bullet C)$                 #2  $X \supset (Y \bullet Z)$                 #3  $A \bullet B$   
       $\therefore \sim F$                          $\therefore X \supset Y$                          $\therefore B$

Number 1 is an invalid argument. If we knew that John did not pass both the final and the course, we could not be sure that it was the final that he did not pass. Recall from Chapter 7 that  $\sim(F \bullet C)$  is the same in meaning as  $\sim F \vee \sim C$ , and from  $\sim F \vee \sim C$  we surely cannot conclude  $\sim F$ . So note that the negation is the major connective of the premise in number 1, and simplification must have a conjunction as a major connective in the premise. Number 2 is valid, but the major connective in the premise is a conditional ( $\supset$ ) rather than a conjunction. We will learn a way in the next chapter to prove #2 to be valid using a combination of rules. Number 3 is valid and has a conjunction as a major connective in the premise, but the conclusion does not match the first part of the conjunctive premise. We will also be learning a way to prove this obviously valid argument by using simplification in combination with another rule.

Numbers 1 and 2 reveal a rule about all the rules of inference: **The rules of inference apply to whole lines only**. Simplification cannot be applied if only a part of a premise is a conjunction; the major connective must be a conjunction. Similarly, given the premise  $A \supset (B \vee C)$ , and the premise  $\sim B$ , we cannot conclude  $C$  by the disjunctive syllogism rule. The  $B \vee C$  is part of a line and must be a whole line by itself for the rule to apply.

$A \supset (B \vee C)$   
 $\sim B \therefore C$  (invalid and not DS)

## ADDITION (Add)

Our last rule of inference is called addition. It is an elementary argument that is valid due to the commitment we have already made to the meaning of the inclusive *or*. Recall that if one part of a disjunction is true, then the entire statement is true. So if a conclusion has (**v**) as a major connective, then the conclusion would have to be true if the first part of the disjunction is true. So it would be impossible for a premise to be true, which is the first part of a disjunctive conclusion, and that disjunctive conclusion false. Here are some examples and the argument form.

| Addition       |                              |                                        |                                            |
|----------------|------------------------------|----------------------------------------|--------------------------------------------|
| Egs.           | #1 A                         | #2 $\sim X$                            | #3 $A \equiv \sim B$                       |
|                | $\therefore A \vee B$        | $\therefore \sim X \vee (C \supset D)$ | $\therefore (A \equiv \sim B) \vee \sim D$ |
| Argument Form: | $p$<br>$\therefore p \vee q$ |                                        |                                            |

Note the essence of the addition rule. There is always a single premise and that premise can be anything. The major connective of the conclusion is always a disjunction and the premise is the first part of the disjunction.<sup>8</sup> What is added to the premise by a disjunction can be anything. Here is an example.

Suppose I am walking down a street with a friend who is a doctor. We come across a person lying flat out on the ground. His breathing is quite labored and he looks terrible. The color of his skin is not right. I say to my doctor friend, "This guy looks very drunk." My doctor friend says, "Either that or he is suffering insulin shock (the symptoms are the same)."

D  
 $\therefore D \vee I$

Do not confuse the "addition" name of this rule with a conjunction. Concluding **p•q** from just **p** is invalid. If we knew John passed the course, we would know that he passed the course *or* the final. But we would not know that he passed the course *and* the final.

---

<sup>8</sup> Obviously an argument would be valid that had the second part of the disjunction of the conclusion as a premise (**q**). However, since we are interested in identifying one elementary argument at a time, we will not classify this as addition. We will have a way of proving the argument form **q**  $\therefore p \vee q$  to be valid in Chapter 10.

We would also know that he passed the final *or* the moon is made of green cheese! This is not as counter-intuitive as it may seem, because we clearly would not know that he passed the final *and* the moon is made of green cheese.

This concludes the introduction to the nine rules of inference. We will now test your pattern recognition ability with some name-the-rule-exercises. Attempting to apply each rule is no different than walking into a room and recognizing the difference between a chair and a table. You must notice that a particular object fits the pattern we call *chair* before you sit down. This is a very complex neurological activity that we take for granted. Those who work in the field of artificial intelligence have learned that getting a computer to recognize a chair is not easy. Some fairly complex computer programs do fine until the chair is turned upside down. Yet a normal human being has little trouble walking into a room, seeing a strange chair turned upside down, turning it right side up, and then sitting on it.

In some of the following exercises you may have a similar experience to that of the confused computer. Just as you have learned to be flexible in applying the pattern of a chair to strange instances of chairs that are not right side up, so you will need to be flexible in applying the nine rules. For instance, the following are examples of modus ponens, modus tollens, and hypothetical syllogism respectively.

|    |                                      |    |                                          |    |                                           |
|----|--------------------------------------|----|------------------------------------------|----|-------------------------------------------|
| #1 | $A$                                  | #2 | $\sim(A \vee B)$                         | #3 | $\sim Z \supset A$                        |
|    | $A \supset \sim C \therefore \sim C$ |    | $X \supset (A \vee B) \therefore \sim X$ |    | $B \supset \sim Z \therefore B \supset A$ |
|    | <b>MP</b>                            |    | <b>MT</b>                                |    | <b>HS</b>                                 |

These examples show that there is nothing absolute about the order of the premises for those rules that have two premises. Just as you learned in grammar school that there was nothing absolute about the addition order of a rule like " $3 + 2 = 5$ ." You eventually (you have probably forgotten the original confusion and trauma) learned that both

$$\begin{array}{r} 3 \\ +2 \\ \hline 5 \end{array} \qquad \begin{array}{r} 2 \\ +3 \\ \hline 5 \end{array}$$

were applications of the rule. So even though the premises are not presented in the same order as the rule was originally presented, all three examples above fit the essential features of their respective rule. As in all examples of modus ponens, number 1 has a conditional premise ( $\supset$ ), another premise that matches the antecedent of the conditional premise, and then a conclusion that matches the consequent of the conditional premise. As in all examples of modus tollens, number 2 has a conditional premise ( $\supset$ ), another premise that negates the consequent of the conditional premise, and then a conclusion

that is the negation of the antecedent of the conditional premise. As in all examples of hypothetical syllogism, number 3 has two conditional premises ( $\supset$ ) and a conditional conclusion, the consequent of one premise matches the antecedent of the other premise ( $\sim Z$ ), and the antecedent of one of the conditional premises (B) is linked with the consequent of the other conditional premise (A) in the conclusion.

In other words, the presentation of these rules using variables could have been

|                                              |                                                   |                                                        |
|----------------------------------------------|---------------------------------------------------|--------------------------------------------------------|
| <b>p</b>                                     | <b><math>\sim q</math></b>                        | <b><math>q \supset r</math></b>                        |
| <b><math>p \supset q \therefore q</math></b> | <b><math>p \supset q \therefore \sim p</math></b> | <b><math>p \supset q \therefore p \supset r</math></b> |
| MP                                           | MT                                                | HS                                                     |

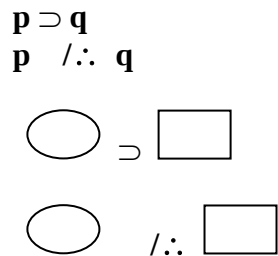
This flexibility in application applies to all the rules that have two premises. But there is no similar flexibility for rules with only one premise. For instance, it would not be an example of the addition rule to have  $A \vee B$  as a premise and conclude A from this. If we knew that either Lisa or Jorge is coming to the party, we would not know for sure that it was Lisa who was coming. The nine rules of inference "move" in only one direction.

$L \vee J$   
 $\therefore L$  (invalid and not Addition)

*Suggestion:* Before doing the following exercises, on a separate sheet of paper you should write down each of the translated examples, the *form* of each rule in variables, and a representation of the form in shapes as demonstrated below with modus ponens.

#### MODUS PONENS (MP)

|                  |                  |                                      |
|------------------|------------------|--------------------------------------|
| #1 $A \supset B$ | #2 $R \supset J$ | #3 $\sim(I \equiv R) \supset \sim P$ |
| $A \therefore B$ | $R \therefore J$ | $\sim(I \equiv R) \therefore \sim P$ |





## Step 1 Exercises

For each of the following arguments state, if applicable, the argument form that justifies the argument as valid. If the argument is not one of the nine rules of inference, indicate this with an X.

$$1. (S \vee E) \bullet (A \vee G) \\ \therefore S \vee E$$

$$2. (A \bullet B) \supset C \\ \therefore (A \bullet B) \supset [(A \bullet B) \bullet C]$$

$$3. * B \supset C \\ \therefore (B \supset C) \vee (B \supset C)$$

$$4. G \supset H \\ \sim H \therefore \sim G$$

$$5. \sim(A \bullet C) \supset (D \vee E) \\ \sim(A \bullet C) \therefore D \vee E$$

$$6. \sim(D \bullet K) \bullet (L \vee Y) \\ \therefore \sim(D \bullet K)$$

$$7. A \supset (B \bullet C) \\ \therefore A \supset B$$

$$8. (S \bullet T) \vee [(U \bullet V) \vee (U \bullet W)] \\ \sim(S \bullet T) \therefore (U \bullet V) \vee (U \bullet W)$$

$$9. O \supset P \\ \sim O \therefore \sim P$$

$$10. (T \bullet U) \vee (C \bullet B) \\ T \bullet U \therefore C \bullet B$$

$$11. [N \supset (O \vee P)] \bullet [Q \supset (O \bullet R)] \\ N \vee Q \therefore (O \vee P) \vee (O \bullet R)$$

$$12. * (X \vee Y) \supset \sim(Z \bullet A) \\ \sim\sim(Z \bullet A) \therefore \sim(X \vee Y)$$

$$13. F \supset (G \bullet D) \\ (G \bullet D) \supset \sim X \\ \therefore F \supset \sim X$$

$$14. \sim(A \supset I) \supset (A \equiv I) \\ (I \equiv H) \supset \sim(A \supset I) \\ \therefore (I \equiv H) \supset (A \equiv I)$$

$$\begin{array}{l}
 15. (C \supset K) \vee (P \supset L) \\
 \quad L \supset M \\
 \quad \therefore [(C \supset K) \vee (P \supset L)] \bullet (L \supset M)
 \end{array}$$

$$\begin{array}{l}
 16.* \sim A \vee (D \bullet B) \\
 \quad \sim A \quad \therefore D \bullet B
 \end{array}$$

$$\begin{array}{l}
 17. A \supset B \\
 \quad \therefore (A \supset B) \vee C
 \end{array}$$

$$\begin{array}{l}
 18. [(A \bullet B) \vee C] \supset D \\
 \quad (A \bullet B) \vee C \quad \therefore D
 \end{array}$$

$$\begin{array}{l}
 19. (A \vee B) \supset C \\
 \quad C \quad \therefore A \vee B
 \end{array}$$

$$\begin{array}{l}
 20. [E \supset (F \supset G)] \vee (C \vee D) \\
 \quad \sim[E \supset (F \supset G)] \quad \therefore C \vee D
 \end{array}$$

$$\begin{array}{l}
 21. C \supset (D \vee G) \\
 \quad (D \vee C) \supset (H \bullet T) \\
 \quad \therefore C \supset (H \bullet T)
 \end{array}$$

$$\begin{array}{l}
 22. (X \supset T) \bullet (H \supset P) \\
 \quad X \vee H \quad \therefore T \vee P
 \end{array}$$

$$\begin{array}{l}
 23.* P \vee T \\
 \quad (P \supset \sim X) \bullet (T \supset \sim Y) \\
 \quad \therefore \sim X \vee \sim Y
 \end{array}$$

$$\begin{array}{l}
 24. (X \bullet P) \bullet (T \supset P) \\
 \quad \therefore X \bullet P
 \end{array}$$

$$\begin{array}{l}
 25. H \supset T \\
 \quad (H \supset T) \supset \sim X \\
 \quad \therefore \sim X
 \end{array}$$

$$\begin{array}{l}
 26. \sim(N \bullet P) \\
 \quad T \supset (N \bullet P) \\
 \quad \therefore \sim T
 \end{array}$$

$$\begin{array}{l}
 27.* (A \bullet B) \\
 \quad (X \bullet P) \supset (A \bullet B) \\
 \quad \therefore X \bullet P
 \end{array}$$

$$\begin{array}{l}
 28. (H \bullet T) \vee (S \supset P) \\
 \quad \sim(H \bullet T) \\
 \quad \therefore S \supset P
 \end{array}$$

$$\begin{array}{l}
 29. A \vee \sim(C \equiv D) \\
 \quad [A \vee \sim(C \equiv D)] \supset \sim\sim Z \\
 \quad \therefore \sim\sim Z
 \end{array}$$

$$\begin{array}{l}
 30. \sim\sim(I \bullet Z) \\
 \quad \sim(I \bullet Z) \vee [H \equiv \sim(P \bullet V)] \\
 \quad \therefore H \equiv \sim(P \bullet V)
 \end{array}$$

## ***Strategies for Pattern Recognition***

Before we continue with the next step in learning formal proofs, we should stop and reflect on some strategies that you may have unconsciously used in identifying the answers to Step 1.

From one point of view, we can see that three of the rules of inference conclude with a match of a part of a premise. Modus ponens, disjunctive syllogism, and simplification all have part of one of the premises as a conclusion. From another point of view, we can see that many of the rules of inference always have the same major connective in the conclusion. Modus tollens will always have a negation ( $\sim$ ) as the major connective of the conclusion. Absorption and hypothetical syllogism will always have a conditional ( $\supset$ ) as a conclusion. Addition and constructive dilemma will always have a disjunction ( $\vee$ ) as a conclusion. And the conjunction rule will always have a conjunction ( $\bullet$ ) as the major connective of its conclusion.

Because a large part of pattern recognition is simply the ability to "stare" at the right features at the right time, we can use the insights just mentioned to develop strategies of staring. When confronted with a complex problem,

### **Strategy 1**

See if the conclusion is a part of a premise, and then try to match the problem with either modus ponens, disjunctive syllogism, or simplification.

Or, if Strategy 1 does not work:

### **Strategy 2**

Focus on the connective of the conclusion, and then try to match with an appropriate rule. If the major connective of the conclusion is a negation ( $\sim$ ), try to match with modus tollens; if the major connective of the conclusion is a conditional ( $\supset$ ), try to match with either absorption or hypothetical syllogism; if the major connective in the conclusion is a disjunction ( $\vee$ ), try to match with addition or constructive dilemma; and, if the major connective is a conjunction, try to match with the conjunction rule ( $\bullet$ ).

For an example, recall the super messy looking argument (A3) from Chapter 8.

1.  $\{[(A \supset B) \equiv (C \vee \sim B)] \bullet \sim D\} \supset (\sim E \vee \sim F)$
2.  $\{[(A \supset B) \equiv (C \vee \sim B)] \bullet \sim D\} \quad \therefore (\sim E \vee \sim F)$

There is so much that we could look at in this argument. It is easy to suffer sensory overload and be overwhelmed by all the detail. Many students panic when looking at a problem with this much detail. Where do we start? There is so much detail and nine possible rules to look at for a possible match. However, anyone should at least be able to stay calm enough to notice that this problem has two premises, so it could not possibly be an example of any of our rules with only one premise. This simple insight immediately cuts down on the amount of staring we need to do. We don't have to look at simplification, absorption, or addition.

Next we use the above strategies. In spite of this problem's complexity we see that strategy 1 applies -- the conclusion  $\sim E \vee \sim F$  is a part of one of the premises. This cuts down on our staring even more, because now we will focus only on modus ponens and disjunctive syllogism. (Remember that simplification was already eliminated because it has only one premise.) Next we notice that the major connective of the premise with  $\sim E \vee \sim F$  has a conditional as the major connective. So at this point we should be staring at only modus ponens.

It is important to reflect that at this point our strategy has enabled us to follow a trail by eliminating possibilities and arrive at a *hypothesis* that this problem may be an example of modus ponens. The rest of the parts still have to fit. The second premise must match the antecedent. If it did not, then we would have to try something else. But here the strategy works because the second premise matches exactly what we need.

Let's see how strategy 2 would be applied when strategy 1 fails. Number 14 was one of the hardest problems in the above exercises.

$$14. \sim(A \supset I) \supset (A \equiv I) \\ (I \equiv H) \supset \sim(A \supset I) \quad \therefore (I \equiv H) \supset (A \equiv I)$$

Here is how to stay disciplined and calm applying the strategies. First we notice that there are two premises. This immediately eliminates simplification, absorption, and addition. Next we notice that the conclusion is not a self-contained part of any premise. The antecedent of the conclusion,  $(I \equiv H)$ , and the consequent of the conclusion,  $(A \equiv I)$ , are themselves parts of premises, but the entire conclusion,  $(I \equiv H) \supset (A \equiv I)$ , is not a part of any premise. This eliminates strategy 1 and the rules modus ponens, disjunctive syllogism, and simplification from consideration. So using strategy 2 we focus on the major connective of the conclusion. Since the major connective is a conditional ( $\supset$ ) we now have a hypothesis that the answer might be hypothetical syllogism. (Absorption also has a conditional as a major connective for its conclusion, but this rule was already eliminated as a possibility because it has only one premise.) We now check the rest of the problem for a pattern match with hypothetical syllogism. Is the major connective of

both premises a conditional ( $\supset$ )? Yes. Does one of the consequents of the premise conditionals match one of the antecedents? Yes,  $\sim(A \supset I)$  is a consequent of the second premise and an antecedent of the first premise. Is an antecedent of one of the premises linked with a consequent of the other premise in the conclusion? Yes,  $(I \equiv H)$ , the antecedent of the second premise, is linked with  $(A \equiv I)$ , the consequent of the first premise. All the parts fit the pattern, hence this is an example of HS.

Some students will still not see HS fit number 14 because even though they suspect it is HS, they will forget that the order of the premises does not matter. They will try to fit HS on 14 like this:

$$\begin{array}{l} p \supset q \\ q \supset r \quad \therefore p \supset r \end{array}$$

Good try, but the fit is like this:

$$\begin{array}{l} q \supset r \\ p \supset q \quad \therefore p \supset r \end{array}$$

$$\begin{array}{l} p = (I \equiv H) \\ q = \sim(A \supset I) \\ r = (A \equiv I) \end{array}$$

Each strategy by itself will not always work. Combined, however, they provide a way of focusing and disciplining our attention when faced with confusing details and eliminating possibilities in steps. As noted previously in this book, without some logical strategies our minds are like a radio whose channel selector is moving back and forth chaotically, never quite focusing on a particular channel long enough so that only confused noise is received. The Internet in some ways has not helped us in this regard. Some educational experts worry that reading and focusing abilities wane when people don't really read web pages but hypertext and surf their way through pages clicking link after link. Reading this book slowly and using logical strategies of problem solving will help your brain get some focusing ability back.

In Step 2 of learning how to construct formal proofs we will be increasing the amount of "noise" that you will need to filter into coherent channels of pattern recognition, and the above strategies will be very helpful. Resist the temptation to jump around looking for some easy fix. At least at first, try to use the strategies step by step. Don't use the "stare and hope" method. It just increases anxiety. After using the strategies a few times, if you can do step 1 problems quickly, then you should begin to just see connections begin to "pop out."

## ***Step 2: Justifying Reasoning Trails with the Rules of Inference***

Much of learning involves an initial step of mimicking. Painters often learn to imitate the styles of previous masters before they develop their own style. Musicians study and play the works of the great composers before they compose their own music. In this section you will study formal proofs that have already been completed. The reasoning trail will be presented, but the premises and the rules used to justify each line will be omitted. Your job will be that of a detective. You will need to reconstruct the thought processes in terms of premises and rules used to create the trail of reasoning. Let's use our original formal proof as an example.

1.  $I \supset (C \vee K)$
2.  $I$
3.  $\sim C$
4.  $K \supset (\sim S \bullet E)$
5.  $M \supset S \quad \therefore \sim M$
6.  $C \vee K$
7.  $K$
8.  $\sim S \bullet E$
9.  $\sim S$
10.  $\sim M$

First recall what you are looking at. The first five lines are premises. The  $\sim M$  is the conclusion. Lines 6 through 10 represent the valid reasoning trail to show that  $\sim M$  follows validly from these premises.

Because our focus is to provide a justification for the reasoning trail (lines 6-10), we start with line 6. We know that line 6 is a conclusion from some line or combination of lines of the premises 1-5, but how do we find this justification when there is so much detail to consider? We use the strategies discovered in the last section.

Starting with strategy 1, we ask whether line 6 is a part of one of the premises above. Right away we have a possible connection: Line 6 is part of premise 1. So, we focus on the rules modus ponens, disjunctive syllogism, and simplification. Then, because line 6 is a consequent of a conditional ( $\supset$ ) of premise 1, we should consider modus ponens, because of the three rules for strategy 1, only modus ponens concludes a consequent from a conditional ( $\supset$ ) of a premise. But modus ponens requires two premises and the second premise must be a match of the antecedent. Because the antecedent is  $I$ , we must have an  $I$  by itself as a premise to complete the connection. We have it. Premise 2 is  $I$ , so we

have discovered that premises 1 and 2 fit the pattern of modus ponens and justify the conclusion  $C \vee K$ .

We have:

$$\begin{array}{ll} (1) I \supset (C \vee K) & p \supset q \\ (2) I \quad \therefore (6) C \vee K & p \therefore q \end{array}$$

So, we write this down as the justification for line 6.

1.  $I \supset (C \vee K)$
2.  $I$
3.  $\sim C$
4.  $K \supset (\sim S \bullet E)$
5.  $M \supset S \quad \therefore \sim M$
6.  $C \vee K \quad (1)(2) \text{ MP}$
7.  $K$
8.  $\sim S \bullet E$
9.  $\sim S$
10.  $\sim M$

The next line to justify is line 7. Using strategy 1, we see that  $K$  occurs as a part of lines 1, 4, and 6. In such situations you must know the rules well. You must know by thoroughly practicing Step 1 exercises that no rule of inference would pull out a  $K$  from a premise like  $I \supset (C \vee K)$ . No rule allows us to conclude part of a consequent. Similarly, no valid rule concludes the antecedent of a conditional as in line 4. (The fallacy of affirming the consequent concludes with the antecedent of a conditional, but this form should never be used in a formal proof of validity.) Thus, only line 6 remains as a possibility for applying strategy 1. Focusing on line 6 then, because the connective is a disjunction ( $\vee$ ) and the  $K$  is in the right location, disjunctive syllogism is the best hypothesis. But disjunctive syllogism uses two premises and to confirm our hypothesis we must find a  $\sim C$ . Premise 3 is  $\sim C$ , so we have discovered that line 7 is justified by lines 6 and 3 and the rule of disjunctive syllogism.

We have found:

$$\begin{array}{ll} (6) C \vee K & p \vee q \\ (3) \sim C \quad \therefore K & \sim p \therefore q \end{array}$$

Our proof now looks like this:

1.  $I \supset (C \vee K)$
2.  $I$
3.  $\sim C$
4.  $K \supset (\sim S \bullet E)$
5.  $M \supset S \quad \therefore \sim M$
6.  $C \vee K$  (1)(2) MP
7.  $K$  (6)(3) DS<sup>9</sup>
8.  $\sim S \bullet E$
9.  $\sim S$
10.  $\sim M$

Next line 8. Again we see that strategy 1 looks promising. The  $\sim S \bullet E$  of line 8 is a part of line 4. It is also in the right location within a conditional statement ( $\supset$ ) to be a modus ponens, which means that to complete the match we will need to find the antecedent  $K$  as a line by itself. The line we just completed is  $K$ , so we have a match in lines 4 and 7 and the rule modus ponens as a justification for line 8. So, we have found:

- |                                     |                  |
|-------------------------------------|------------------|
| (4) $K \supset (\sim S \bullet E)$  | $p \supset q$    |
| (7) $K \therefore \sim S \bullet E$ | $p \therefore q$ |

Our proof will now look like this:

1.  $I \supset (C \vee K)$
2.  $I$
3.  $\sim C$
4.  $K \supset (\sim S \bullet E)$
5.  $M \supset S \quad \therefore \sim M$
6.  $C \vee K$  (1)(2) MP
7.  $K$  (6)(3) DS
8.  $\sim S \bullet E$  (4)(7) MP
9.  $\sim S$
10.  $\sim M$

Perhaps you have noticed another strategy that we could combine with strategies 1 and 2.

Because we are justifying a chain of reasoning, there is a high probability that the line we just completed will be used to get the next line. Line 6 was used to get line 7, and

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<sup>9</sup> It does not matter what order the lines are referred to; we could have also written (3)(6) DS.

line 7 was used to get line 8. Like all our strategies we will see that this strategy is only a rule of thumb -- it will not always work -- but as another technique for focusing our attention systematically it will be very useful. Let's try it now. Our next line is 9. Checking line 8, we see that strategy 1 works again. The  $\sim S$  is a part of line 8 and since this line is a conjunction ( $\bullet$ ) and the  $\sim S$  is in the appropriate location, the simplification rule applies. So, now we have found:

|                         |                |
|-------------------------|----------------|
| (8) $\sim S \bullet E$  | $p \bullet q$  |
| $\therefore (9) \sim S$ | $\therefore p$ |

Our proof will now look like this:

|                                   |                     |
|-----------------------------------|---------------------|
| 1. $I \supset (C \vee K)$         |                     |
| 2. $I$                            |                     |
| 3. $\sim C$                       |                     |
| 4. $K \supset (\sim S \bullet E)$ |                     |
| 5. $M \supset S$                  | $\therefore \sim M$ |
| 6. $C \vee K$                     | (1)(2) MP           |
| 7. $K$                            | (6)(3) DS           |
| 8. $\sim S \bullet E$             | (4)(7) MP           |
| 9. $\sim S$                       | (8) Simp.           |
| 10. $\sim M$                      |                     |

Line 10 completes our proof, but this time strategy 1 fails. The  $\sim M$  does not occur as a part of any premise.<sup>10</sup> So we must now shift our mental gears to strategy 2 and focus on the connective of the line we want to justify. The connective in line 10 is a negation ( $\sim$ ).

The only rule that always has a negation as the major connective of the conclusion is modus tollens. Applying this strategy requires a thorough understanding of the rules. In this case we must be able to reconstruct what the premise would be for a conclusion  $\sim M$  by the rule modus tollens. Since modus tollens concludes the negation of the antecedent of a conditional ( $\supset$ ), we need a premise like  $M \supset \_\_\_$  and then the negation of the consequent. Line 5 has an  $M$  as an antecedent of a conditional, and the line we just completed,  $\sim S$ , negates the consequent. So we have discovered that line 10 is justified by lines 5 and 9 and the rule of modus tollens.

|                                     |                            |
|-------------------------------------|----------------------------|
| (5) $M \supset S$                   | $p \supset q$              |
| (9) $\sim S \therefore (10) \sim M$ | $\sim q \therefore \sim p$ |

Our complete proof will now look like this:

---

<sup>10</sup> Remember that the  $\sim M$  adjacent to line 5 is not a premise. It is the conclusion as indicated by  $\therefore$  and only shows us the end point of our proof. So it is never used as a line of justification in a proof.

1.  $I \supset (C \vee K)$
2.  $I$
3.  $\sim C$
4.  $K \supset (\sim S \bullet E)$
5.  $M \supset S \quad \therefore \sim M$
6.  $C \vee K$  (1)(2) MP
7.  $K$  (6)(3) DS
8.  $\sim S \bullet E$  (4)(7) MP
9.  $\sim S$  (8) Simp.
10.  $\sim M$  (5)(9) MT

A word of warning before you try some exercises. You must be flexible in applying the strategies. When a strategy does not work, you must give it up. In working on line 7 above, we noticed that the  $K$  is a part of line 4. But no matter how hard we try and how long we stare, line 4 cannot be made to work with any of our rules. So strategy 1 fails for this line and we had to stop staring at line 4 and move on. Often students will continue to stare at line 4 as they move on to strategy 2! They will be trying to get strategy 2 to work while looking at a line that failed for strategy 1. Like giving up a belief that does not work, it is often hard to give up anchoring your attention on a failed hypothesis. This is why a playful attitude of trial and error using a method of hypothesis and test is most appropriate for finding justifications. Your attitude should be experimental but focused.

Similar remarks apply to strategy 2. Suppose we had the following proof and we were working on line 7.

1.  $P \supset Y$
2.  $(X \bullet Y) \supset \sim H$
3.  $D \supset [P \supset (X \bullet Y)]$
4.  $D$
5.  $\sim H \supset X \quad \therefore (P \supset X) \bullet (P \supset Y)$
6.  $P \supset (X \bullet Y)$  (3)(4) MP
7.  $P \supset \sim H$
8.  $P \supset X$
9.  $(P \supset X) \bullet (P \supset Y)$

Since line 7 is not part of any premise, we shift to strategy 2. Since line 7 is a conditional ( $\supset$ ), either absorption or hypothetical syllogism must apply. Absorption cannot apply because line 7 does not have a consequent that contains a conjunction. So focusing on hypothetical syllogism we try to reconstruct the premises that would give us  $P \supset \sim H$  as a conclusion. We could use some scratch paper and set up our staring as follows:

$P \supset$   
 $\_ \supset \sim H \therefore P \supset \sim H$

We then notice that line 1 is  $P \supset Y$ , a possibility for the first premise of the hypothetical syllogism. But to match we would need a second premise of  $Y \supset \sim H$ , and there is no such line anywhere above line 7.

(1)  $P \supset Y$   
 (?)  $Y \supset \sim H \therefore P \supset \sim H$

What gives? Strategy 1 did not work and now strategy 2 also seems to be failing. In the Step 2 exercises that follow there will be no "bogus" steps. A correct justification will exist for each line. We must be flexible enough to try strategy 2 again. Since premise 1 ( $P \supset Y$ ) did not work, we must try another premise that is a conditional and has P as an antecedent. Premise 6 is a possibility, so we set up a possible scenario for hypothetical syllogism using this premise as follows:

(6)  $P \supset (X \bullet Y)$   
 (??)  $\_ \supset \sim H \therefore$  (7)  $P \supset \sim H$

Then looking for a match to complete the hypothetical syllogism we see that the second premise,  $(X \bullet Y) \supset \sim H$ , in our proof completes the match. We found:

(6)  $P \supset (X \bullet Y)$                        $p \supset q$   
 (2)  $(X \bullet Y) \supset \sim H \therefore P \supset \sim H$                        $q \supset r \therefore p \supset r$

Keep these flexibility points in mind as you work on the following exercises.



## Step 2 Exercises

State the justification for each line that is not a premise of the following arguments.

#1

1.  $Z \bullet A$
2.  $(Z \vee B) \supset C \therefore Z \bullet C$
3.  $Z$
4.  $Z \vee B$
5.  $C$
6.  $Z \bullet C$

#3\*

1.  $(D \vee G) \bullet (H \vee I)$
2.  $(D \supset H) \bullet (G \supset I)$
3.  $\sim H \therefore I$
4.  $D \vee G$
5.  $H \vee I$
6.  $I$

#5

1.  $\sim R \supset S$
2.  $\sim T \supset (U \supset V)$
3.  $T \vee (\sim R \vee U)$
4.  $\sim T \therefore S \vee V$
5.  $U \supset V$
6.  $(\sim R \supset S) \bullet (U \supset V)$
7.  $\sim R \vee U$
8.  $S \vee V$

#2

1.  $K \supset (B \vee I)$
2.  $K$
3.  $\sim B$
4.  $I \supset (\sim T \bullet N)$
5.  $N \supset T \therefore \sim N$
6.  $B \vee I$
7.  $I$
8.  $\sim T \bullet N$
9.  $\sim T$
10.  $\sim N$

#4

1.  $H \supset I$
2.  $I \supset J$
3.  $K \supset L$
4.  $H \vee K \therefore J \vee L$
5.  $H \supset J$
6.  $(H \supset J) \bullet (K \supset L)$
7.  $J \vee L$

#6

1.  $O \supset P$
2.  $(O \bullet P) \supset Q$
3.  $\sim(O \bullet Q) \therefore \sim O$
4.  $O \supset (O \bullet P)$
5.  $O \supset Q$
6.  $O \supset (O \bullet Q)$
7.  $\sim O$

#7

1.  $X \supset Y$
2.  $(X \supset Z) \supset (A \vee Y)$
3.  $(X \bullet Y) \supset Z$
4.  $\sim A \therefore Y$
5.  $X \supset (X \bullet Y)$
6.  $X \supset Z$
7.  $A \vee Y$
8.  $Y$

#9

1.  $G \supset \sim H$
2.  $\sim G \supset (I \supset \sim H)$
3.  $(\sim J \vee \sim I) \supset \sim \sim H$
4.  $\sim J \therefore \sim I$
5.  $\sim J \vee \sim I$
6.  $\sim \sim H$
7.  $\sim G$
8.  $I \supset \sim H$
9.  $\sim I$

#8\*

1.  $(\sim A \vee B) \supset D$
2.  $(D \vee B) \supset [\sim A \supset (C \equiv E)]$
3.  $\sim A \bullet D \therefore C \equiv E$
4.  $\sim A$
5.  $\sim A \vee B$
6.  $D$
7.  $D \vee B$
8.  $\sim A \supset (C \equiv E)$
9.  $C \equiv E$

#10

1.  $A \supset B$
2.  $\sim(A \bullet B)$
3.  $A \vee (\sim \sim L \bullet \sim \sim K)$
4.  $P \supset \sim L \therefore \sim P \vee \sim Y$
5.  $A \supset (A \bullet B)$
6.  $\sim A$
7.  $\sim \sim L \bullet \sim \sim K$
8.  $\sim \sim L$
9.  $\sim P$
10.  $\sim P \vee \sim Y$

### ***Step 3: On Your Own, Constructing Formal Proofs with the Rules of Inference***

The purpose of Step 1 and 2 exercises was to build up your own pattern recognition ability to the point that you can construct your own formal proofs. In the next set of exercises you will be faced with problems like the following.

1.  $A \supset B$
2.  $B \supset C$
3.  $\sim C \therefore \sim A$

There will be nothing but empty space under the last premise. Your task will be to put down valid steps, to create your own chain of reasoning until you arrive at the conclusion. Like life, you must create your own trail into an uncertain future. Also like life, not all of the trails created will lead to the desired conclusion, and even those that

do, some will be better (more elegant) than others. Consider two solutions to the problem of fixing the flat tire of a car that is parked on a steep hill. The car cannot simply be jacked up, because the steepness of the hill and gravity will cause the car to slip off the jack. Person one sees that there is a large sturdy tree not too far from the parked car, so some rope is borrowed to tie the car securely to the tree so it will not slide towards the bottom of the hill. But none of the pieces of rope that this person has are long enough to reach the tree from the parked car. (Assume also that the pieces tied together do not reach.) So this person gets the “bright” idea of putting a shopping cart between the car and the tree, such that one piece of rope can reach from the car to the shopping cart and another from the shopping cart to the tree. For further stability, this person gets some heavy bricks and puts them in the shopping cart.

Consider person two passing by and watching person one struggling with the exertion of moving the heavy bricks and trying to make the ropes as tight as possible so the car will not roll off the jack. Person two sees another car parked a few feet downhill of the car with a flat tire and gets the owner to simply back this car up against the car with the flat tire. They leave a very small space between the bumpers, and then securely fasten the emergency brake and turn the wheels into the curb. The solution of person one may have worked, but surely the solution of person two is more elegant. It is simpler, faster, requires less effort, and fewer things can go wrong.

From one point of view, the goal of life is not only to solve our problems, but to do so elegantly like that of person two. Many people “bump” into the world like person one.<sup>11</sup> They survive, but the solutions they have for their problems lack grace and simplicity, and these solutions are often a little scatterbrained, involving lots of wasted energy. In constructing formal proofs for the first time, many logic students will be like person one. They will blunder forward and perhaps arrive at the desired conclusion. But their proofs will be a unnecessarily complicated.

However, as an initial learning strategy, blundering forward is exactly what most students need to do. At least it is better than just staring aimlessly, putting down no steps at all, or putting down steps that are not valid. Applied to formal proofs, a strategy of **blundering forward** means: In looking at the above argument do not worry about the conclusion; focus on the premises and ask yourself if you recognize any premise or combination of premises that would fit a rule of inference, such that the premise or combination of premises would entitle you to create a conclusion, a beginning line in a proof. If you see a fit that entitles you to create a conclusion, put it down as a step and don't worry yet whether you are on the right trail that will lead to the final conclusion. Your strategy is that if you put down enough steps, you will eventually blunder across the conclusion.

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<sup>11</sup> Big corporations also. Many viewed with dismay BP's Rube Goldberg 2010 efforts to cap the Deep Water Horizon oil leak in the Gulf of Mexico.

Following this strategy, in looking at the above premises your pattern recognition ability should be strong enough by now to see that premises 1 and 2 fit a pattern of hypothetical syllogism.<sup>12</sup> So we would be entitled to create the step  $A \supset C$ . Since we are blundering forward in just trying to create any steps we see, we can put this step down as follows:

1.  $A \supset B$
2.  $B \supset C$
3.  $\sim C \therefore \sim A$
4.  $A \supset C$  (1)(2) HS

Remember that we get to create steps (conclusions) now if premises fit a rule.

- |                                                |                                      |
|------------------------------------------------|--------------------------------------|
| (1) $A \supset B$                              | $p \supset q$                        |
| (2) $B \supset C \therefore$ (4) $A \supset C$ | $q \supset r \therefore p \supset r$ |

We may or may not be on the right track. We don't worry about it yet. Taking a look at premises 2 and 3, we notice another pattern, modus tollens. Should we do it? Why not? We are just trying to get down steps at this point. Premises 2 and 3 entitle us to create the step  $\sim B$ . And we could have even done this step first.

- |                                      |                            |
|--------------------------------------|----------------------------|
| (2) $B \supset C$                    | $p \supset q$              |
| (3) $\sim C \therefore$ (5) $\sim B$ | $\sim q \therefore \sim p$ |

If we do both of these steps, our proof would look like this:

1.  $A \supset B$
2.  $B \supset C$
3.  $\sim C \therefore \sim A$
4.  $A \supset C$  (1)(2) HS
5.  $\sim B$  (2)(3) MT

See anything else to do? Remember that lines 4 and 5 can now be used also to create more steps. Most people can see several ways to create the conclusion at this point. Sometimes, however, even the most obvious connections right in front of us are hard to see if we are a little nervous or confused. I once had a mature female student who was returning to college after raising a family. She was very nervous and intimidated by the fact that she was competing with much younger students and much younger minds. (Because she was nervous and intimidated, she studied and did very well.) I noticed that

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<sup>12</sup> If you do not see that premises 1 and 2 fit the pattern of hypothetical syllogism, you should re-do Step 1 exercises. If you can't do Step 1 quickly, do it again, otherwise you will be wasting your time trying to do Step 3.

every time she did a formal proof, she would take the first two premises and apply the rule of conjunction, no matter what the conclusion or level of difficulty for the proof. She would then finish the proof in a most elegant way, and her conjunction step was seldom needed. So I finally asked her one day why she did proofs this way. She answered, "To relax! I know that I can always take any two premises and combine them by conjunction, and I just need to get started and get rid of some of that blank space, then I'm ok." Well she was right; there is nothing wrong with her conjunction step from the strict point of view of valid applications of the rules. A conjunction can be done at any time. So another valid step that could be done would be to take premises 1 and 2 and create a conjunction.

|                                                                  |                            |
|------------------------------------------------------------------|----------------------------|
| (1) $A \supset B$                                                | $p$                        |
| (2) $B \supset C \therefore (A \supset B) \bullet (B \supset C)$ | $q \therefore p \bullet q$ |

Remember that from the point of view of the strategy of blundering forward all we are trying to do is create valid steps. So, if we did this step also, our proof would now look like this:

|                                          |              |
|------------------------------------------|--------------|
| 1. $A \supset B$                         |              |
| 2. $B \supset C$                         |              |
| 3. $\sim C \therefore \sim A$            |              |
| 4. $A \supset C$                         | (1)(2) HS    |
| 5. $\sim B$                              | (2)(3) MT    |
| 6. $(A \supset B) \bullet (B \supset C)$ | (1)(2) Conj. |

We could continue to blunder like this for many more steps, but at this point it should be apparent that we don't need to. Either the combination of lines (1) and (5) or (4) and (3) by modus tollens would produce  $\sim A$ . For instance,

|                                |                            |
|--------------------------------|----------------------------|
| (4) $A \supset C$              | $p \supset q$              |
| (3) $\sim C \therefore \sim A$ | $\sim q \therefore \sim p$ |

Here then would be our finished proof:

|                                          |              |
|------------------------------------------|--------------|
| 1. $A \supset B$                         |              |
| 2. $B \supset C$                         |              |
| 3. $\sim C \therefore \sim A$            |              |
| 4. $A \supset C$                         | (1)(2) HS    |
| 5. $\sim B$                              | (2)(3) MT    |
| 6. $(A \supset B) \bullet (B \supset C)$ | (1)(2) Conj. |
| 7. $\sim A$                              | (4)(3) MT    |

Complete but not very elegant. Although there is nothing wrong with the above proof from the strict point of view of the rules of inference (each step is a valid application of one of the rules), a much more elegant proof exists. Like the solution of person one above, there are extra steps that are not needed. But we are just learning to feel our way in creating symbolic reasoning trails, so the goal is to put down a trail and worry about its elegance later. Gradually, as your recognition ability increases you will learn other strategies for making proofs more elegant.

We can show you one now. Once you are very comfortable with the rules, you can try the strategy of **working backwards**. Unlike the blundering forward method where the conclusion is ignored at first, the working backwards method requires that we focus on the conclusion and ask a hypothetical question: Given the premises, which premise is most likely to be involved in the last step of a proof for the desired conclusion, and what other step would need to be created that combined with this premise would give us the conclusion? Since our conclusion is  $\sim A$ , premise 1 looks promising. Since premise 1 is  $A \supset B$ , we know that IF we find a  $\sim B$ , then we would be entitled to conclude a  $\sim A$  by modus tollens.

|                                |                            |
|--------------------------------|----------------------------|
| (1) $A \supset B$              | $p \supset q$              |
| (?) $\sim B \therefore \sim A$ | $\sim q \therefore \sim p$ |

We don't have a  $\sim B$  yet. So we ask the same question again but now directed at finding a  $\sim B$ . Given our premises, what premise might be involved in creating a  $\sim B$  and what other step would we need to create the conclusion of a  $\sim B$  as a valid step? Premise 2 looks promising. Since premise 2 is  $B \supset C$ , we know that IF we find a  $\sim C$  we can create a  $\sim B$  again by the rule of modus tollens.

|                                |                            |
|--------------------------------|----------------------------|
| (2) $B \supset C$              | $p \supset q$              |
| (?) $\sim C \therefore \sim B$ | $\sim q \therefore \sim p$ |

If we did not have the  $\sim C$  anywhere, we would continue the what-if searching. But we are done. We do not need to create a  $\sim C$  because this is premise 3. This is the goal of working backwards: To work backwards by asking what-if questions until the trail ends in a premise that we already have. Here is a symbolic picture of our reasoning.

|      |               |    |
|------|---------------|----|
| Want | $\sim A$      |    |
| Have | $A \supset B$ |    |
| Need | $\sim B$      | MT |
| Want | $\sim B$      |    |
| Have | $B \supset C$ |    |

|      |          |                       |
|------|----------|-----------------------|
| Need | $\sim C$ | MT                    |
| Want | $\sim C$ | Already have $\sim C$ |

This process can be mapped out on scratch paper. To reconstruct this backward reasoning into the forward reasoning of a formal proof, follow the advice of the children's TV show *Sesame Street*, where children are taught that if you are lost make everything that was last, first, and you will find your way home. What we wanted last was a  $\sim C$ , but we already have this step in premise 3. What we wanted next to last was a  $\sim B$ , so this will be the first step to put down in our proof. We needed  $\sim B$  so we could create what we wanted first, a  $\sim A$ . So our next, and in this case last, step in our proof will be  $\sim A$ . Here is how this proof will look placed along side our original proof using the method of working backwards.

### Working Backwards

1.  $A \supset B$
2.  $B \supset C$
3.  $\sim C \therefore \sim A$
4.  $\sim B$  (2)(3) MT
5.  $\sim A$  (1)(4) MT

### Blundering Forwards

- |                                          |              |
|------------------------------------------|--------------|
| 1. $A \supset B$                         |              |
| 2. $B \supset C$                         |              |
| 3. $\sim C \therefore \sim A$            |              |
| 4. $A \supset C$                         | (1)(2) HS    |
| 5. $\sim B$                              | (2)(3) MT    |
| 6. $(A \supset B) \bullet (B \supset C)$ | (1)(2) Conj. |
| 7. $\sim A$                              | (4)(3) MT    |

The working backwards proof is clearly more elegant. It is shorter and there are no extra steps. But the blundering forward proof is not incorrect. Each step in this proof is a valid application of one of the rules of inference. As in life, note that one can be logical but a little crazy. The mere use of logic and technology does not guarantee elegance and wise applications. Obviously the ultimate goal is to have elegant proofs and reasoning trails. But ideal goals are not always achievable. There were 40 million lines of computer code in the computer programs that ran the initial U.S. space shuttle. Undoubtedly from a divine point of view more elegant program trails existed that could have accomplished the same tasks. And each year as the shuttles flew into space and more experience was gained, the hard working men and women who run the space shuttle program discovered more elegant and efficient reasoning trails to get the job done, but a perfect program was probably elusive for human beings.

In the beginning, do not worry about whether your proof is elegant. More important is to get started on a trail and eventually reach the conclusion with all valid steps. Every step must be a valid application of one of the rules of inference or the proof will be worthless. A single line of invalid computer syntax or logic will cause our technological products, such as the space shuttle, to produce dangerous output. The following is a typical example of student error in producing a formal proof.

- |                                          |                    |
|------------------------------------------|--------------------|
| 1. $A \supset B$                         |                    |
| 2. $B \supset C$                         |                    |
| 3. $\sim C \therefore \sim A$            |                    |
| 4. $A \supset (A \bullet B)$             | (1) Abs.           |
| 5. $A$                                   | (4) Simp. <b>X</b> |
| 6. $(A \supset B) \bullet (B \supset C)$ | (1)(2) Conj.       |
| 7. $B$                                   | (1)(5) MP          |
| 8. $A$                                   | (1)(7) MP <b>X</b> |
| 9. $C$                                   | (2)(7) MP          |
| 10. $C \vee \sim A$                      | (9) Add.           |
| 11. $\sim A$                             | (10)(3) DS         |

There are some creative steps in this proof, such as the series of steps 9 through 11. Unfortunately the proof is worthless because steps 5 and 8 are not correct applications of simplification and modus ponens respectively. Step 4 cannot yield **A** by simplification because it has a conditional ( $\supset$ ) as a major connective and simplification must have a conjunction ( $\bullet$ ).

- |                               |                           |                      |
|-------------------------------|---------------------------|----------------------|
| (4) $A \supset (A \bullet B)$ | $p \supset (p \bullet q)$ | $p \bullet q$        |
| $\therefore$ (5) $A$ <b>X</b> | $\therefore p$ ???        | $\therefore p$ Simp. |

Step 8 is the fallacy of affirming the consequent, not modus ponens.

- |                                     |                      |                     |
|-------------------------------------|----------------------|---------------------|
| (1) $A \supset B$                   | $p \supset q$        | $p \supset q$       |
| (7) $B \therefore$ (8) $A$ <b>X</b> | $q \therefore p$ ??? | $p \therefore q$ MP |

In the exercises that follow you should produce as many steps as you possibly can, but you should check your steps periodically to see if each is a correct application of a rule of inference. Your basic goal should be to produce a valid reasoning trail and avoid any **X**'s. Then, if you are successful in deriving the conclusion, inspect your proof for superfluous steps to see if you could rewrite your proof in a more elegant way. There is not necessarily only one elegant way to do a proof. Like life there may be many ways to accomplish a goal. As an illustration of this point, note that the first problem in the following exercises is the same problem we have been working on. See if you can create an elegant, two-line proof that is different than the working backwards proof above.



### Step 3 Exercises

Construct a formal proof of validity for each of the following arguments. (For number 1, produce a two-line proof that is different than the working backwards example above.)

#1

1.  $A \supset B$
2.  $B \supset C$
3.  $\sim C \therefore \sim A$

#2

1.  $A \supset B$
2.  $A \vee C$
3.  $\sim B \therefore C$

#3

1.  $D \supset B$
2.  $F \vee \sim B$
3.  $\sim F \therefore \sim D$

#4\*

1.  $G \supset H$
2.  $I \supset D$
3.  $G \vee I \therefore H \vee D$

#5

1.  $\sim D$
2.  $\sim D \supset \sim B$
3.  $C \supset B \therefore \sim C$

#6

1.  $J \supset (K \bullet L)$
2.  $S \vee J$
3.  $\sim S$
4.  $K \supset (S \vee T) \therefore T$

#7\*

1.  $S \supset L$
2.  $(H \bullet T) \supset (P \bullet X)$
3.  $L \supset (H \bullet T)$
4.  $\sim(P \bullet X)$
5.  $\sim S \supset (A \supset B) \therefore A \supset (A \bullet B)$

#8

1.  $A \supset B$
2.  $C \supset D$
3.  $A \vee C$
4.  $\sim B$
5.  $D \supset (S \supset B) \therefore \sim S$

#9

1.  $(D \equiv \sim H) \supset (P \vee \sim T)$
2.  $\sim \sim X$
3.  $D \equiv \sim H$
4.  $P \supset \sim X$
5.  $\sim T \supset Y$
6.  $(\sim X \vee Y) \supset G \therefore G \bullet \sim P$

#10<sup>13</sup>

1.  $X \supset (Y \supset H)$
2.  $X$
3.  $H \supset P$
4.  $Y$
5.  $\sim(H \bullet P) \therefore Q$

## Translations and Formal Proofs

Translate the following arguments into symbolic notation. Check your answers with your instructor, then provide formal proofs for each translation.

1. If we buy the new car, then we will not have enough money for basic necessities provided we also pay for car insurance. If we buy the new car, we'll have to have car insurance. We are buying the new car. So, we will not have enough money for basic necessities.

C = We buy the new car.

B = We have enough money for basic necessities.

I = We pay for car insurance.

2. If all cars had air bags, car insurance premiums would go down for the following reasons: If all cars had air bags, hundreds of thousands of crippling injuries would be eliminated each year. The elimination of thousands of crippling injuries each year is a sufficient condition for the lowering of medical insurance premiums. The lowering of medical insurance premiums will cause car insurance premiums to go down.

A = All cars had air bags.

C = Car insurance premiums would go down.

I = Hundreds of thousands of crippling injuries would be eliminated.

M = Medical insurance premiums will be lowered.

3. Johnson must have contracted AIDS in prison or our theory on AIDS needs revision.

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<sup>13</sup> **HINT:** Beginning this proof is not difficult; ending it is more challenging. A contradiction lurks in the premises of this argument. The proof cannot be solved unless you find the contradiction. See the discussion on contradictions in the Brief Truth Tables section of Chapter 8. Then once you find the contradiction, you have to put two and two together so to speak to use the contradiction to get Q.

This is so, because the incubation period for AIDS is approximately ten years. Johnson has been free for only two years since completing his twenty year sentence. Now if the incubation period for AIDS is approximately ten years, then provided Johnson has been free for only two years since completing his twenty year sentence, he must have contracted AIDS in prison.

I = The incubation period for AIDS is approximately ten years.

F = Johnson has been free for only two years since completing his twenty year sentence.

P = Johnson must have contracted AIDS in prison.

R = Our theory on AIDS needs revision.

- 4.\* Necessary conditions for solving our country's drug problem involve not only reducing the supply of drugs, but also having effective drug treatment programs and an effective anti-drug education program. If we continue to follow the administration's program, which to date has involved spending over 60 billion dollars primarily on reducing the supply of drugs, we will not address all the necessary conditions. We are following the administration's program. Hence, either we don't solve our country's drug problem or we don't continue to follow the administration's program.

S = We solve our country's drug problem.

R = We reduce (or attempt to reduce) the supply of drugs.

T = We have effective drug treatment programs.

E = We have an effective anti-drug education program.

A = We follow (or continue to follow) the administration's program

5. If Vietnam falls to communism, then Cambodia falls. If Vietnam and Cambodia fall, then Laos falls. If Vietnam, Cambodia, and Laos fall, then Thailand falls. If all of the above fall, then all of Southeast Asia falls. Therefore, if Vietnam falls, all of Southeast Asia falls. (Note: The formal proof of this argument is challenging.)

V = Vietnam falls to communism.

C = Cambodia falls to communism.

L = Laos fall to communism.

T = Thailand falls to communism.

A = All Southeast Asia fall to communism.

6. If either taxes are raised again or oil prices rise, then the economy will not continue to improve. Either the economy continues to improve or the Republicans will have a potentially decisive political issue in November. If the Republicans will have a potentially decisive political issue in November and the Democrats do not have a counter issue of concern to the American people, then control of the White House will change. We know taxes are being raised again. We also know that the Democrats do not have a

counter issue of concern to the American people. Accordingly, either control of the White House will change or the Republicans will not have a potentially decisive political issue in November.

T = Taxes are raised again.

O = Oil prices rise.

E = The economy will continue to improve.

R = The Republicans will have a potentially decisive political issue in November.

D = The Democrats do have a counter issue of concern to the American people.

W = Control of the White House will change.

7. Passing algebra with a C grade or better is a necessary condition for Cisa to be eligible to take calculus. Moreover, passing calculus is a necessary condition for Cisa to take engineering physics. Cisa has not passed algebra with a C grade or better. Since Cisa obviously can't pass calculus, if she is not eligible to take calculus, it follows that Cisa cannot take engineering physics.

A = Cisa passes algebra with a C grade or better.

E = Cisa is eligible to take calculus.

C = Cisa passes calculus.

P = Cisa takes engineering physics.

8. If Spock is emotional, then he is like doctor McCoy. On the other hand, if Spock is logical, then he has a predominant Vulcan personality. Either Spock is emotional or logical.<sup>14</sup> Everyone knows that Spock is not like doctor McCoy. So, Spock has a predominant Vulcan personality.

E = Spock is emotional.

M = Spock is like doctor McCoy.

L = Spock is logical.

V = Spock has a predominant Vulcan personality.

9. It is not true that having all true premises is a necessary condition for being able to have a valid argument. If it is not true that having all true premises is a necessary condition for being able to have a valid argument, then one is able to have a valid argument even though that argument has false premises. Now, although having all true premises is a necessary condition for having a sound argument, it is not true that having all true premises is a sufficient condition for a valid or sound argument. If having all true

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<sup>14</sup> The intention of this *or* statement is exclusive. According to this book, this premise is a questionable dilemma.

premises is a necessary condition for having a sound argument, then it is not possible to have both a sound argument and one with false premises. Hence, although one is able to have a valid argument and have false premises, one cannot both have a sound argument and one with false premises.

T = One has an argument with all true premises.

V = One is able to have a valid argument.

S = One has a sound argument.

Translate "One has an argument with false premises" as "One does not have an argument with all true premises."

10. The student predicament of our times?

If I take the forty-hour per week job and take three classes per semester, then I will let the quality of my family life suffer. If I continue to progress at a sustained pace toward my degree and support my family, then I need to take the forty hour per week job and take three classes per semester. If I don't both continue to progress at a standard pace toward my degree and support my family, then I will support neither. If I can neither continue to progress at a sustained pace toward my degree nor support my family, then I will not be happy. I will not let the quality of my family life suffer. However, if I don't continue to progress at a sustained pace toward my degree, I will not be eligible for a scholarship. So, either I am not going to be happy and not eligible for a scholarship, or I will need to win the state lottery or readjust my priorities.

F = I take the forty-hour per week job.

C = I take three classes per semester.

Q = I will let the quality of my family life suffer.

P = I continue to progress at a standard pace toward my degree.

S = I support my family.

H = I will be happy.

E = I will be eligible for a scholarship.

L = I will need to win the state lottery.

R = I will need to readjust my priorities.

Note: To allow for a formal proof using the nine rules, translate neither P nor S using the  $\sim P \bullet \sim S$  version.

## Answers to Starred Exercises

### Step 1

$$\begin{array}{ll} 3. B \supset C & p \\ \therefore (B \supset C) \vee (B \supset C) & \therefore p \vee q \end{array}$$

**Add**

Remember that **q** as a variable can stand for **any** simple or compound statement.

$$\begin{array}{ll} 12. (X \vee Y) \supset \sim(Z \bullet A) & p \supset q \\ \sim\sim(Z \bullet A) \therefore \sim(X \vee Y) & \sim q \therefore \sim p \\ \text{MT} & \end{array}$$

$$\begin{array}{ll} 16. \sim A \vee (D \bullet B) & p \vee q \\ \sim A \therefore D \bullet B & p \therefore q ?? \end{array}$$

**X**

This is a very common student mistake. Note that the second premise should be  $\sim\sim A$  to stay true to the rule of DS.

$$\begin{array}{ll} 23. P \vee T & p \vee r \\ (P \supset \sim X) \bullet (T \supset \sim Y) & (p \supset q) \bullet (r \supset s) \\ \therefore \sim X \vee \sim Y & \therefore q \vee s \\ \text{CD} & \end{array}$$

$$\begin{array}{ll} 27. (A \bullet B) & q \\ (X \bullet P) \supset (A \bullet B) & p \supset q \\ \therefore X \bullet P & \therefore p \end{array}$$

**X**

Fallacy of Affirming the Consequent

### Step 2

#3

$$\begin{array}{ll} 1. (D \vee G) \bullet (H \vee I) & \\ 2. (D \supset H) \bullet (G \supset I) & \\ 3. \sim H \therefore I & \\ 4. D \vee G & (1) \text{Simp} \\ 5. H \vee I & (2)(4) \text{CD} \end{array}$$

6. I (3)(5) DS

#8

1.  $(\sim A \vee B) \supset D$
2.  $(D \vee B) \supset [\sim A \supset (C \equiv E)]$
3.  $\sim A \bullet D \therefore C \equiv E$
4.  $\sim A$  (3) Simp.
5.  $\sim A \vee B$  (4) Add
6.  $D$  (1)(5) MP
7.  $D \vee B$  (6) Add
8.  $\sim A \supset (C \equiv E)$  (7)(2) MP
9.  $C \equiv E$  (4)(8) MP

### Step 3

#4

1.  $G \supset H$
2.  $I \supset D$
3.  $G \vee I \therefore H \vee D$
4.  $(G \supset H) \bullet (I \supset D)$  (1)(2) Conj.
5.  $H \vee D$  (4)(3) CD

Note that the conjunction play works in this proof! In fact, I often get a proof from students that concludes line 5 right away as line 4, justified by (1)(2)(3) CD. This is impossible because CD does not have 3 premises (none of our rules do). But this is an interesting mistake, because it shows that the student who is making this mistake is beginning to see more than one step at a time. We will often have quick insights like this, but then we must "unpack" them. That is, the wholes that we are intuitively seeing must be broken down into parts. Supposedly, Mozart was able to see his entire 40th symphony in a split second of inspiration. It then took him days to write it out, detail by detail. This symphony takes about a half hour to play!

#7

1.  $S \supset L$
2.  $(H \bullet T) \supset (P \bullet X)$
3.  $L \supset (H \bullet T)$
4.  $\sim(P \bullet X)$
5.  $\sim S \supset (A \supset B) \therefore A \supset (A \bullet B)$

- |                               |           |
|-------------------------------|-----------|
| 6. $S \supset (H \bullet T)$  | (1)(3) HS |
| 7. $\sim(H \bullet T)$        | (2)(4) MT |
| 8. $\sim S$                   | (6)(7) MT |
| 9. $A \supset B$              | (5)(8) MP |
| 10. $A \supset (A \bullet B)$ | (9) Abs.  |

Note: This proof can be done several different ways with the same number of steps.

### Translations and Formal Proofs

- 4.
- |                                              |                                 |
|----------------------------------------------|---------------------------------|
| 1. $S \supset [R \bullet (T \bullet E)]$     |                                 |
| 2. $A \supset \sim[R \bullet (T \bullet E)]$ |                                 |
| 3. $A$                                       | $\therefore \sim S \vee \sim A$ |
| 4. $\sim[R \bullet (T \bullet E)]$           | (2)(3) MP                       |
| 5. $\sim S$                                  | (1)(4) MT                       |
| 6. $\sim S \vee \sim A$                      | (5) Add                         |

**Essential Logic**

Ronald C. Pine